PLANE CURVES OF THE THIRD ORDER

BY

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PREFACE.

The plane curves of the third order have four features in which they chiefly differ from those of the second order. As the equation of a cubic curve contains four more constants than a quadric, a larger group of transformations is appropriate if not essential; the projective group is employed instead of the group of affinity and rigid motions. While the projective group has eight parameters, the cubic curve has nine, and there is one invariant or characteristic constant exhibited by every proper i. e. not composite cubic. This different group and the single absolute invariant are the first novel feature. The second is the possibility of two real branches, and the certainty that one of them is *odd*, whereas the conic can have but a single real branch, and that an *even* one. The third is the occurrence of *points* of *inflexion*, where the tangent line has three in place of two coincident points of intersection. The system of the nine inflexions is doubtless, geometrically, the central and conspicious part of the theory of these curves. The fourth feature is seen in the so called parameter-representation of the points of a non-singular cubic, — one whose odd branch does not cross itself; — for while rational or trigonometric functions of the parameter sufficed for conics, and indeed for the very extensive class of *rational curves*, a cubic requires elliptic functions of the parameter.

Closely connected with this last is the fact that on a given cubic not all the points can be selected arbitrarily in which the curve shall be intersected by another algebraic curve. One point is fixed by the choice of the others. This is the origin of the theory of residuation, whose logical basis is the famous "Fundamental theorem" of Noether. It is impossible, since 1870, to ignore the need of this theorem, yet no standard text book has to my knowledge presented a strict proof; it is plausible, but requires demonstration. Here will be found

a proof simplified as far as possible by restrictive hypotheses; and it is believed that careful reading of this argument will give the student an insight into more complicated cases.

As to invariants, a minimum of purely formal work is introduced. The polar process and the Hessian determinant are shown to be unchanged, save for factors, by linear transformations. Then by the use of a normal form the number of constants is reduced, and the H. C. F. process shows the Hessian of a Hessian linearly related to the fundamental cubic and its Hessian in an identity, whose coefficients are necessarily invariants. This seems a strategic and quick way to reach the two formal invariants, and the one absolute. From these, by a development in Taylor's series the two evectants or line-cubics of principal importance for geometry are found, and so the doubly apolar line-cubic. This development may interest by its novelty those to whom the theory of invariants is well known. The fifth chapter culminates in the division of proper cubics into three classes by the discriminant.

In addition to the more usual modes of generation of cubic curves, there will be found that by semi-perspective involutions, employed so effectively in Schroeter's synthetic treatise; and the pentagon construction of P. Serret.

For the elliptic integral of the first kind, the Weierstrassian normal equation of the curve is used as a matter of course; and for demonstration of Abel's theorem on complete sets of points a discussion from the Algebra of H. Weber, the most direct elementary proof that I have seen, was readily adaptable. This is equivalent to the addition-theorem for the Weierstrassian p-function, and this geometrical setting has much to commend it, at any rate in contrast to the often preferred treatment of sinam (u).

Though this book is intended as merely an introduction to a rich and attractive field, (in some sense a complement to the sections on cubics in *Pascal's Repertorium* and in *G. Kohn's* synopsis in the *Encyklopädie der mathematischen Wissenschaften*, it seemed imperative that the elements of apolarity should be included. On the other hand the subject

of linear systems of cubics in two or more parameters is so broad that a brief chapter is hardly possible. Other minor omissions it is hoped will be pardoned on the plea of brevity. The author sincerely hopes that no reader will be satisfied with this meagre introduction, but that is may be found helpful as a stepping-stone to many extensive and beautiful treatises on special themes, and a stimulus to further exploration.

It is believed that the treatment of sets of nine associated points of a plane is somewhat more full and satisfactory, i. e. more rigorous, here than in any text hitherto published. The construction of the ninth point when eight are given is one of the special problems most frequently attacked in mathematical journals, and it may still be matter of doubt what solution is most economical of labor and logic. While chief stress is laid on analysis, I have thought it almost essential on this problem to outline the practical solution indicated by the equations; but I have not examined here the expedients for lessening labor invented by the various writers, not even the solution by Rohn, singled out for commendation by Kohn in the Encyklopädie.

The author desires here to acknowledge with gratitude the assistance of three institutions: Northwestern University and its Acting President, Daniel Bonbright, who granted a semester's leave of absence when this work was begun; Vassar College and the National Research Council, who jointly assumed financial risk; also the generous cooperation of the Harvard University Press, which undertakes the publication.

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CHAPTER I.

The real branches of a cubic.

§ 1. Definition of a cubic, proper or degenerate. Real and imaginary points. Projective transformations.

A plane curve of third order, or cubic curve, is the locus of any equation of the third degree in rectilinear coordinates. The equation may be non-homogeneous, that is, it may contain two variables to the third and lower degrees; or it may be made homogeneous by the use of trilinear coordinates, and then every term will be of the third degree. Under this definition three right lines must be considered as constituting together a cubic curve; so also a conic and one right line, since their equations can be so combined by multiplication as to form one of third degree satisfied by all points of the three lines, or of the line and the conic. For if $l = 0$, $m = 0$, $n = 0$ are equations of three lines, then the equation $l \cdot m \cdot n = 0$ is satisfied for all points of each line. The aggregate of a conic: $\Phi = 0$, and a line: $l = 0$, is the locus of the cubic equation: $l \cdot \Phi = 0$. These we shall call degenerate cubics, defining the term thus. *A cubic which contains all the points of a curve of order lower than the third is a degenerate cubic; all others are proper cubics.*

Denote by $f(x, y, 1) = 0$ any real non-homogeneous cubic equation in rectilinear coordinates x and y. For any real value of x this equation has as roots either three real values of y, or one real and two imaginary values. Again for real y, x may be imaginary; or the equation may be satisfied by imaginary values of both coordinates. If, of two values (x, y) satisfying a cubic equation $f(x, y, 1) = 0$, either one or both are imaginary, they are said to be coordinates of an imaginary point of the cubic curve. Every curve will possess imaginary points, since imaginary values may be assumed for either coordinate, and the resulting equation must give corresponding values for the other. Certainly also every

cubic in whose equation the coefficients are all real must contain real points, since to a real value x corresponds at least one real root of the cubic equation for y. These real points may be so connected that a variable point can pass from any one to any other without leaving the locus of real points; in that case the locus is said to consist of one real branch. Otherwise there may be two real branches of a cubic, both together constituting the entire real cubic. The first important question presented is this: *What is the form, or what are all the possible forms of real cubics?** ↖

The discussion of this question is much simplified if we adopt the projective point of view. We shall therefore consider as equivalent forms all cubics which are projective to one another. From the projective standpoint,† all infinitely distant points are said to lie on a straight line, called the line at infinity. The plane can be so transformed by projection that the line at infinity shall be brought into any arbitrarily chosen position in the finite region; conversely, any arbitrary line may be projected to infinity. In fact, any four lines may be projected into the position of any other four, provided that no three lines in either set have a point in common. Every such projective transformation, or collineation, is represented algebraically by some linear substitution upon homogeneous rectilinear coordinates. For present purposes projective transformation will be used only to transfer infinitely distant points to the finite region in a plane, and vice versa.

§ 2. Intersections of a right line and a real cubic.

A real cubic is met by every real line in either one or three real points. Let the cubic be given by the equation $f(x, y, 1) = 0$. If the equation of the line be $y = ax + b$, then points common to both must satisfy the equation of degree 3 in x,

$$f(x, ax + b, 1) = 0. \tag{1}$$

* I. Newton: Enumeratio linearum tertii ordinis (1706). Moebius: Ueber die Grundformen der Linien der dritten Ordnung (1849); see his Gesammelte Werke, Bd. II, p. 89. Cayley: On the classification of cubic curves (1864); in Collected Mathematical Papers, Vol. V, p. 354.

† Salmon's Conic Sections, Chapter 17.

As all coefficients in this equation are real, there must be either one or three real roots. But a real x gives a real y, hence there are one or three real points of intersection. Of course it might happen that (1) should be identically satisfied for all values of x, and in that case all points of the line must lie in the cubic,* hence the latter would be degenerate. The above theorem is therefore true without qualification only for proper cubics.

From this it is evident that a real cubic has either one or three points at infinity; that is, intersections with the line at infinity. This is true even of degenerate cubics given by non-homogeneous equations; since if such a cubic contains the entire line at infinity its equation cannot be of higher degree than the second. For example, an ellipse and a right line constitute a cubic having one real point at infinity, while a hyperbola and any secant line form a cubic with three distinct real points at infinity, and a parabola together with one diameter constitute a cubic with three coincident infinite points.

§ 3. Odd and even branches. Two classes of real cubics.

Denote by A a real point of a proper cubic. Suppose a variable point P to start from the position A, moving along the real cubic always in the same sense, that is, not reversing at any point the direction of its motion. Eventually P must return to A, for an algebraic curve cannot have an end-point, either in the finite region,† or at infinity if we regard it projectively. If P passes through all the real points of the cubic in going from A to A always in the same sense, the cubic is said to have a single real branch; but if there are real points not traversed by P, the cubic must have at least two real branches. *A proper cubic cannot have more than two real branches.* Assume any two real points A, B on one branch, and join them by a right line. This line must cut the cubic in a third real point C. If C is on the

* See also Chapter II, § 1.

† See Byerly's Differential Calculus, p. 247.

same branch with A and B, this is termed an odd branch, because every line must meet it in an odd number of points; for if the line ABC is moved continuously until it coincides with any arbitrary line, no one point of intersection can pass to another branch except by passing through imaginary points (§ 2); and imaginary intersections necessarily occur in pairs, so that every line meets this odd branch either in three points or in one point. But if C is on a second branch of the curve, the first is called an even branch, the second an odd branch, for similar reasons. Now a cubic cannot have two even branches, since in that case a line joining a point A on the one and a point C on the other would cut the first in at least one more point B, the second in another point D, or at least four in all, while we have seen (§ 2) that only three intersections are possible. For the same reason the curve cannot have two odd branches; a line joining two points A, B on the one would meet that branch in a third point C, the other branch in at least one point B, or at least four in all. So far then as odd and even branches are concerned, a real cubic must fall into one of two classes: those with only one branch, an odd one; and those with two branches, one odd and one even.

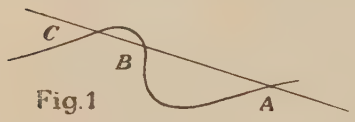


Fig.1

§ 4. Actual examples of unipartite and bipartite cubics.

Are the two possible classes actually represented by real cubics? To see that they are, it will be simplest to use rectangular coordinates. Consider the line and the conic whose equations are respectively: $y = 0$ and $s = x^2 + y^2 - r^2 = 0$. The two loci divide the finite plane into four parts, each characterized by the signs belonging to y and to s, as shown in the diagram (Fig. 2). Assume a point P_0 within the conic, on the negative side of the line, quite near an

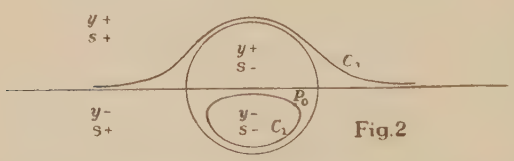


Fig.2

intersection of the two, and let the values in P_0 be $y = y_0$, $s = s_0$. The product is small and positive, say $y_0 s_0 = c$, and $ys = c$ is the equation of a cubic through P. Its locus cannot cross either the line or the conic, but must lie always near to the one or the other, hence its aspect is approximately that of the two branches C_1, C_1 in the above diagram. Again, take a line tangent to the conic, as for example: $y + r = 0$ and $s = x^2 + y^2 - r^2 = 0$. The equation: $(y + r)\, s = + c$ has for locus a cubic curve approximately like that in Fig. 3.

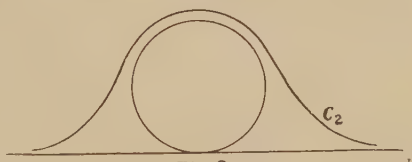


Fig. 3

To vary the figure further, use oblique axes, and consider the two equations: $(y + a)\,(x^2 + y^2 - a^2) = -cx^2$,

$$(y + a)^3 = cx^2.$$

Their loci will be in general like the light curve C_3 and the heavy curve C_4 respectively in Fig. 4. Of the above types C_1 is bipartite, the other three unipartite. C_1 and C_2 are non-singular, while the other two are singular cubics, C_3 having an ordinary double point, and C_4 a cusp. If a cubic has a singular point on a real branch, it can have no other real branch.

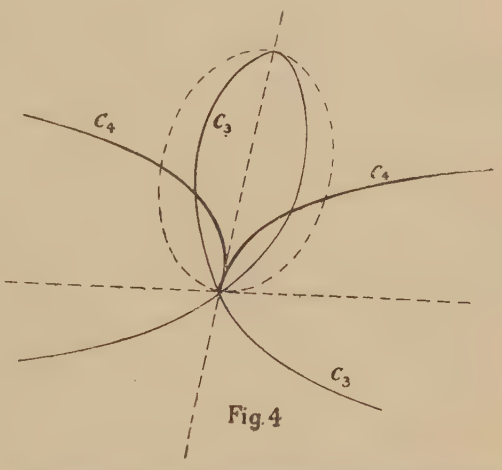


Fig. 4

§ 5. Points of inflexion cannot occur on an even branch, but three or one must occur on an odd branch.

In figures 2 and 3 are seen parts of the odd branch convex downward, and adjoining them a part convex upward. The point of transition between two parts of a branch with con-

vexities on opposite sides is called a point of inflexion. A more precise definition is the following, illustrated in Fig. 5. If the tangent at any point P, on a real branch of a curve, and a secant through the same point contain in opposite angles two arcs of that branch meeting in P, the point P is called a *point of inflexion*. But if the tangent of any point and any secant through the same point contain in two adjacent angles two arcs of the same real branch of the cubic meeting in that point, the point is an *ordinary point* of the cubic. This definition presupposes that the point P is not a double point or a cusp. We infer immediately that projection can neither transform an ordinary point into a point of inflexion, nor vice versa, for by projection vertical or opposite angles are necessarily transformed into opposite angles. The definition further is not applicable to imaginary points of inflexion, for which an algebraic definition must be given.

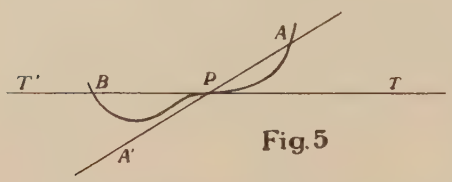


Fig. 5

An even branch of a real cubic cannot contain a point of inflexion. On an even branch let P be a point of inflexion. Then a secant can be drawn through P containing two other points of the same branch. For of two secants connecting P with points B and C on the arcs in opposite angles TPA, $T'PA'$, either PB or PC must lie between TPT' and the other secant. That is, either PT and PB separate the lines PC and PA, or else PT and PC separate PB from PA. In the former alternative (as in the diagram Fig. 6.) the line PC must cross in some point C' the arc PB, hence the line PC must contain a point on each arc, different from P, as C and C'; and as this branch is even, PC must meet it also in a fourth point, beside meeting the odd branch in (at least) one point. But no line can meet a

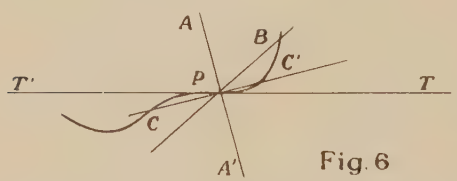


Fig. 6

cubic in five points (§ 2), hence the hypothesis of a point of inflexion on an even branch is absurd.

An odd branch of a real cubic must have one point of inflexion; and must have three unless it has also a double point or a cusp. First, the existence of at least one inflexion is shown as follows. If Q is a point on the odd branch, not an inflexion, double point, or cusp, draw QA tangent to the curve in Q, meeting the odd branch as it must in some additional point A (Fig. 7). If A is a point at infinity, transform by projection so that one arc QA becomes a finite arc,* and so of necessity the chord QA a finite line. At Q the curve is convex toward the line, but in order to intersect at A it must be concave toward the line at some points between Q and A, hence between Q and A must occur at least one inflexion I, as in Fig. 7. The argument fails if the finite arc QA crosses itself, as in Fig. 7a. In that case a line can be found cutting

Fig. 7

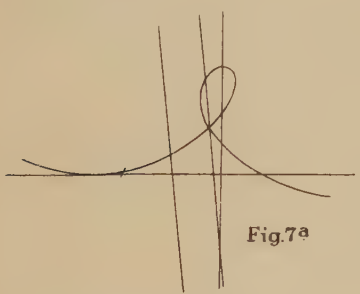


Fig. 7a

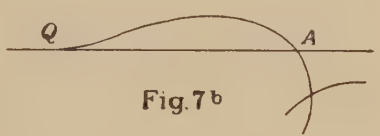


Fig. 7b

the line once, the arc three times, between Q and A; projecting this line to infinity gives a figure like 7b, in which the arc QA is finite and free from double points; for a cubic cannot have two double points, as a line connecting two such

* This is possible if there is only one real point at infinity on the odd branch in question, i. e. if there exists a straight line meeting this branch in one real and two imaginary points, for such a line can be projected to infinity. Such a line can be found by rotating any tangent about one of its points, not the point of contact.

points would meet the cubic in four points. Similarly may be treated the case of a cusp occurring on a finite arc QA.

Second, in the absence of double points and cusps, the existence of three inflexions on the odd branch is proved thus. Take as before for the line at infinity some line cutting the odd branch in only one real point, and that point not an inflexion. In this unique point at infinity, Q, draw the tangent (asymptote), meeting the odd branch in one finite point A. For reasons exactly the same as before, each arc QA now contains at least one inflexion. Or the contact may be at a finite point, and each arc in turn can be made finite; and thus the preceding reasoning applies to show the existence of two inflexions. Finally these two arcs are known to be concave toward the tangent line on opposite sides of the line, hence they are in part of opposite curvature, and a transition point, or third inflexion, must occur. It is to be proved later that no cubic can have more than three *real* inflexions.*

Non-singular real cubics consist therefore of either one odd branch with three real inflexions, or else such an odd branch and an even branch (or oval) with no inflexions. Singular cubics may have a real double point or cusp; and can have then only one branch, an odd one, with one real inflexion.

* This argument is very similar to that of Moebius. See footnote on p. 2.

Chapter II.

How points determine a cubic. Simplest theorems on systems of intersections on a cubic.

§ 1. A proper cubic cannot meet a curve of order n in more than $3n$ points. Real cubics have real equations.

In the theory of conics it is shown that any conic, or locus of an equation of second order, which has more than two points in common with a straight line, contains all points of that line. So of cubics the theorem is true: *If a cubic curve have four points in common with a straight line, it contains all points of that line and is degenerate.*

Proof. Through a fifth point on the straight line, distinct from the four given common points, let a line l be drawn not coinciding with the given line. Project to infinity this line l. Employ non-homogeneous coordinates, taking for the y-axis ($x = 0$) the given line, for x-axis any other finite line not parallel to the first. Separate into two sets the terms in the equation of the cubic, all of the first set containing a factor x, all of the second set not containing x. The equation is then written thus:

$$F_3(x, y, 1) = x \cdot f_2(x, y, 1) + f_3(y, 1) = 0$$

where the subscript denotes the degree of the polynomial. Therefore the intercepts on the line ($x = 0$) must satisfy an equation $f_3(y, 1) = 0$. But this can have only three finite roots, whereas by hypothesis there are four. Hence all coefficients of this equation must be zero, and we have identically:

$$F_3(x, y, 1) \equiv x \cdot f_2(x, y, 1).$$

The locus of the equation: $x \cdot f_2(x, y, 1) = 0$, consists of the conic: $f_2(x, y, 1) = 0$, and the entire line: $x = 0$. q. e. d.

A cubic and a conic may have as many as six points in common, for both may be degenerate, consisting of three and of two lines respectively. But a proper cubic cannot contain seven points of even a degenerate conic; for of the two lines of such a conic, one would necessarily contain at least four of the seven points of intersection, whence the cubic must be degenerate, as just now proved. Regarding intersections of cubics and conics it remains therefore to be proved that — *Any cubic curve which contains seven or more points of a proper (nondegenerate) conic must contain all points of that conic; or a proper cubic can intersect a proper conic in no more than six points.*

Proof. Choose suitably as before, if necessary, a line to be projected to infinity, and fix upon axes such that no two of the seven points have equal values of x. Arrange the equations of the two curves in descending powers of y:

$$F_3(x, y, 1) = y^3 \cdot f_0 + y^2 \cdot f_1(x) + y \cdot f_2(x) + f_3(x) = 0,$$
$$\Phi_2(x, y, 1) = y^2 \cdot \varphi_0 + y \cdot \varphi_1(x) + \varphi_2(x) = 0.$$

Eliminate y from these equations by Sylvester's dialytic method, as follows. Multiply the first by y and 1, the second by y^2, y, and 1, in turn. This gives five equations, all satisfied simultaneously for each of the seven points. They all contain linearly the five quantities y^4, y^3, y^2, y, 1, hence the determinant of the five equations must vanish for a value of x equal to the abscissa of any one of the seven points. Call the determinant $R(x)$, i. e. let

$$R(x) = \begin{vmatrix} f_0 & f_1(x) & f_2(x) & f_3(x) & 0 \\ 0 & f_0 & f_1(x) & f_2(x) & f_3(x) \\ \varphi_0 & \varphi_1(x) & \varphi_2(x) & 0 & 0 \\ 0 & \varphi_0 & \varphi_1(x) & \varphi_2(x) & 0 \\ 0 & 0 & \varphi_0 & \varphi_1(x) & \varphi_2(x) \end{vmatrix}.$$

The equation: $R(x) = 0$ is of sixth degree or lower, as appears by inspection. Since it has by hypothesis seven distinct roots, $R(x)$ must vanish identically. Now $R(x)$ is compounded lin-

early from $F_3(x, y, 1)$ and $\Phi_2(x, y, 1)$. To see this, multiply the first four columns of the determinant respectively by y^4, y^3, y^2, y, and add to the last column. Remembering the meaning of $f_0, f_1(x), f_2(x)$, etc., we have the identity:

$$R(x) \equiv \begin{vmatrix} f_0 & f_1 & f_2 & f_3 & y \cdot F \\ 0 & f_0 & f_1 & f_2 & F \\ \varphi_0 & \varphi_1 & \varphi_2 & 0 & y^2 \cdot \Phi \\ 0 & \varphi_0 & \varphi_1 & \varphi_2 & y \cdot \Phi \\ 0 & 0 & \varphi_0 & \varphi_1 & \Phi \end{vmatrix} \equiv M(x, y^1) \cdot F - N(x, y^2) \cdot \Phi \equiv 0$$

where for brevity f_i denotes $f_i(x)$, F denotes $F_3(x, y, 1)$, etc. $M(x, y)$ and $N(x, y)$ are polynomials of degree 4 and 5 or lower in x, but in y of degree only 1 and 2 respectively. Hence by transposition the identity; $M(x, y) \cdot F(x, y, 1) \equiv N(x, y) \cdot \Phi(x, y, 1)$. The second member must therefore contain the factor $M(x, y)$; but by hypothesis: $\Phi(x, y, 1) = 0$ is the equation of a proper conic, i. e. $\Phi(x, y, 1)$ is not separable into factors rational in x, y; and Φ is not itself a factor of M, which is of lower degree in y, therefore $N(x, y)$ must be exactly divisible by $M(x, y)$, with a polynomial (linear) quotient $P(x, y)$. Division gives thus the identity:

$$F(x, y, 1) \equiv P(x, y) \cdot \Phi(x, y, 1).$$

Accordingly the cubic, $F = 0$, contains all points of the conic, $\Phi = 0$. q. e. d.

As the theorem had been proven already for degenerate conics, it is valid for all conics. The same form of proof can be adopted exactly in establishing the following theorems: the theorem for curves of order n depending, in case of degeneration, upon the proof for all orders lower than n.

THEOREM. *If two cubics have* 10 *points in common, and one is a proper cubic, they coincide throughout. If both be degenerate they have in common at least either an entire line or an entire conic.* A consequence of this theorem is that a proper cubic with more than 9 real points must have an equation with exclusively real coefficients. For if some imaginary coefficients

enter, then for all real values of x, y on the curve, the equation falls into two parts, one with real, the other with pure imaginary coefficients; and these two parts must vanish independently.

$$F(x, y) \equiv F'(x, y) + \sqrt{-1} \cdot F''(x, y)$$
$$\therefore F'(x, y) = 0 \text{ and } F''(x, y) = 0.$$

These equations cannot be of higher degree than the third, and have therefore not more than 9 common solutions unless they are identically equal or have a common factor: in the former case the original equation can be divided by a complex constant, leaving all coefficients real; in the latter case the original equation contains the factor common to the parts, and the curve is degenerate. Real cubics have therefore real equations.

THEOREM. *If a curve of order n (or a C_n) have with any proper plane cubic $3n+1$ common points, it is degenerate, consisting of the cubic and a locus of order $n-3$.*

These theorems are here enunciated for curves whose known intersections are all distinct points, and the proof is dependent in each case upon the degree of the polynomial $R(x)$ which results from elimination of one variable between the two equations. This degree of the *resultant* must remain the same when two or more points of intersection vary, approaching as near as we please to a common limiting position. Hence when r points of intersection coincide, however the axes of coordinates may be chosen, the resultant $R(x)$ will have a corresponding r-fold root. *Conversely we adopt a definition of the multiplicity of an intersection which will render the above theorems valid for all cases, however complicated, of coincident points of interesection,—contact,* osculation, hyperosculation, intersections in double points, triple points, cusps, etc.

Definition. When the resultant $R(x)$ of the equations of two curves: $F(x, y) = 0$ and $\Phi(x, y) = 0$, obtained by eliminating y according to Sylvester's dialytic method, does not vanish identically, then a common point of the two curves (x', y') has a multiplicity r if $x = x'$, is at least an r-fold root of

the equation: $R(x) = 0$ *for every possible choice of the coordinate axes, and is not for every coordinate system an* $(r+1)$*-fold root.*

In all theorems concerning intersections of curves, a common point of multiplicity r is to be counted as r common points.

§ 2. Two curves of orders m and n meet in exactly mn points.

The theorems of § 1 are in one sense negative statements. They fix only a superior limit to the number of intersections of two distinct curves. The positive theorem is equally indispensable, which states that this limit is always the actual number. The proof is found in all complete manuals of algebra.* For reference the theorem is here stated without proof.

THEOREM. *Two plane algebraic curves of orders* m *and* n*, which have not an infinite number of common points, intersect in exactly* mn *points real or imaginary, each point of multiplicity* r *being counted as* r *points of intersection.*

§ 3. Nine points determine a cubic containing them, but for any eight points there is one ninth point which leaves the cubic indeterminate.

The cubic curves present features which have no analogue in the theory of conics. Such are odd branches and points of inflexion, discussed in Chapter I. Others relating to sets of intersections of cubics with other algebraic curves are now to be examined.

THEOREM. *If three cubics have 9 common points, their equations are not linearly independent.*

PROOF. Let the equations of the three cubics be denoted by $\varphi = 0$, $\psi = 0$, $\chi = 0$. Form the equation: $\varphi + \lambda\psi = 0$ with the value of λ indeterminate. It denotes a cubic having

* First in BEZOUT: Théorie générale des équations algébriques (1779), § 47, p. 32.

See also: SALMON, Modern Higher Algebra (1885), p. 71; BIERMANN, Elemente der Mathematik (Leipzig 1895) pp. 259–263; NETTO: Vorlesungen über Algebra (1900), Bd. II, p. 355; CHRYSTAL, A Treatise on Algebra (1886), Vol. 1, p. 284.

in common with the cubic: $\chi = 0$ the 9 points of intersection of the two curves: $\varphi = 0$, $\psi = 0$. On the curve: $\chi = 0$ assume any tenth point (x', y') and let $\varphi(x', y') = \varphi'$, $\psi(x', y') = \psi'$. Neither φ' nor ψ' can be equal to zero, since no two of the given cubics have ten common points. Determine λ' so that $\varphi' + \lambda' \psi' = 0$; then the cubic $\varphi + \lambda' \psi = 0$ has 10 points in common with the cubic $\chi = 0$, and therefore coincides with it throughout (II, § 1). The equations can differ only by a multiplicative constant, hence for some value of μ':

$$\varphi + \lambda' \psi \equiv -\mu' \chi, \quad \text{or} \quad \varphi + \lambda' \psi + \mu' \chi \equiv 0.$$

There exists therefore a *linear identical* relation among the three equations, as asserted by the theorem.

Intending now to find how many cubics can pass through 9 arbitrary points, we must first examine this question: *Can three cubics through eight arbitrarily given points have equations linearly independent?* If possible, let the three equations be: $\varphi = 0$, $\psi = 0$, $\chi = 0$, and call the 8 points $A_1, A_2, A_3, \ldots, A_8$. For all values of λ, μ, the equation:

$$\varphi + \lambda \psi + \mu \chi = 0$$

represents a cubic through the same 8 points. The possible relations among the points may be classified as follows.*

1. All 8 points lie on a line: $g = 0$. Then any cubic containing the points must be degenerate (II, § 1). Any conic: $S = 0$ together with the line constitutes a cubic through the points. As a conic has 5 independent constants, the cubic: $g \cdot s = 0$ has the same number, hence there are 6 linearly independent cubics through the given points.

2. If 7 points lie on a line, the other not, there must be 5 linearly independent cubics containing the points.

3. 6 points on a line, the other 2 off the line, will allow 4 linearly independent cubics.

* It is to be understood that if points described on a line happen to coincide, the line joining them is assumed to coincide with the given line or with the tangent to the conic: exceptional cases can be discussed easily by the reader.

4. 5 on a line, the other 3 elsewhere, will allow 3 linearly independent cubics. The 3 points may be collinear, or not.

5. If 4 are on one line, the remaining 4 on another line, these two with an arbitrary third line form a cubic of the required kind, and of these there are 3 linearly independent. If of the second 4, only 3 are collinear, the cubics linearly independent are only 2; so also if no three of the second set are collinear.

6. If no more than 3 lie on any line, let such a line contain A_1, A_2, A_3, with the equation: $g = 0$. The other 5 completely determine a conic: $S = 0$. The hypothesis that these 8 points can be contained in three linearly independent cubics will lead to a contradiction, and is therefore false. For two points may be selected, B_1 on the line but not on the conic, and B_2 not on either. Using the above notation for the three cubics, let φ, ψ, and χ have

$$\text{in } B_1, \text{ the values } \varphi_1,\ \psi_1,\ \chi_1,$$
$$\text{in } B_2, \text{ the values } \varphi_2,\ \psi_2,\ \chi_2.$$

Determine λ' and μ' by the two equations:*

$$\varphi_1 + \lambda'\psi_1 + \mu'\chi_1 = 0,$$
$$\varphi_2 + \lambda'\psi_2 + \mu'\chi_2 = 0.$$

Then the curve: $F = \varphi + \lambda'\psi + \mu'\chi = 0$ must surely contain the points B_1 and B_2. But the curve F has in the line: $g = 0$ four points, A_1, A_2, A_3, B_1, and is therefore degenerate, consisting of this line and some conic. The conic constituent has five points in common with S, and must coincide with it. Therefore we have identically, with constant factor ν',

$$F \quad \varphi + \lambda'\psi + \mu'\chi \quad \nu' \cdot g \cdot S.$$

The point B_2 however, though lying on F, lies by hypothesis neither upon the line g nor upon the conic S; hence the

* After the point B_1 has been chosen, and the 3 cubics $\varphi = 0$, $\psi = 0$, $\chi = 0$, the B_2 can be taken as not lying on any one of the following 3 cubics: $\psi_1\varphi - \varphi_1\psi = 0$, $\chi_1\varphi - \varphi_1\chi = 0$, $\chi_1\psi - \psi_1\chi = 0$. This insures that λ', μ' are determinate.

hypothesis of three linearly independent cubics through the eight given points is absurd.

7. If no 3 of the 8 points lie on any line, then two cases are to be distinguished.

7a. All 8 points may lie on a conic: $S = 0$. Then any three independent linear equations: $g_1 = 0$, $g_2 = 0$, $g_3 = 0$ combined with S give three linearly independent degenerate cubics:

$$g_1 S = 0, \quad g_2 S = 0, \quad g_3 S = 0,$$

but no proper cubic contains the points, since they number more than 6 on a conic.

7b. When 7 lie on a conic, the others elsewhere, there are no proper cubics, and only two linearly independent degenerate cubics through the 8 points. If 6 and no more lie on a conic: $S = 0$, let a line: $g = 0$ join the other two points. Assuming three independent cubics through the 8 points, require the curve:

$$\varphi + \lambda\psi + \mu\chi = 0$$

to contain also a seventh point B_1 on the conic and a point B_2 on neither the line nor the conic, nor on three particular cubics: (See note under *6* above). As in case *6*, this is proved impossible, hence only two cubics are independent.

8. If no 3 lie on any line and no 6 on any conic, take a conic: $S = 0$ through $A_1, A_2, \ldots, A_5$, and a line: $g = 0$ through A_6, A_7. Choose two additional points on the conic, B_1 and B_2, with the proviso of the note cited above. Assuming three independent cubics through $A_1, A_2, \ldots, A_8$, determine the curve:

$$F' \equiv \varphi + \lambda'\psi + \mu'\chi = 0$$

to contain also B_1 and B_2. As seven of these points are on a conic, we have (II, § 1) identically

$$F' \equiv \nu' \cdot g \cdot S \qquad (\nu' \text{ a constant factor}).$$

But this cubic consists of the line g and the conic S, and so does not contain the point A_8, contrary to hypothesis. Hence in this case again the three cubics are not linearly independent.

To sum up, denote by g_i and S_k the hypotheses of i points on a line and k points on a conic respectively: *the only cases in which* 8 *points are contained in three linearly independent cubics are indicated as follows*:

$$g_8,\ g_7,\ g_6,\ g_5,\ g_4 + g_4,\ S_8.$$

The cubics in all these cases are necessarily degenerate.

Return now to the question concerning cubics through 9 given points. Consider only those sets of 9 points which can lie upon proper cubics: that is, exclude sets which have more than 3 points on any one line, or more than 6 on any one conic. Then it is certain that none of the six exceptional sets of 8 points just now studied will occur. Through 8 of the 9 given points suppose (if possible) two cubics to pass, whose equations are: $\varphi = 0$ and $\psi = 0$. Every third cubic through the same 8 points must have, according to the above theorem an equation of the form: $\varphi + \lambda\psi = 0$. Hence it must contain all intersections of the two curves: $\varphi = 0$ and $\psi = 0$. But those are 9 in number (II, § 2). This fact suggests a theorem so important that we now inquire strictly: How many points on a cubic must be given in order to determine its equation?

The terms of a cubic equation can be arranged in descending powers of y:

$$F(x, y) \equiv y^3 \cdot f_0 + y^2 \cdot f_1(x) + y \cdot f_2(x) + f_3(x) = 0,$$
$$f_0 = A,\ f_2(x) = C_0x^2 + C_1x + C_2,$$
$$f_1(x) = B_0x + B_1,\ f_3(x) = D_0x^3 + D_1x^2 + D_2x + D_3,$$

and the locus is not altered if we assume some one of the constants as equal to unity, for the others can be altered if necessary proportionately. There remain $2 + 3 + 4 - 9$ arbitrary constants in the equation. To require the cubic to contain any one point $(x_0,\ y_0)$ imposes a single condition:

$$F(x_0,\ y_0) = 0 = Ay_0^3 + B_0x_0y_0^2 + B_1y_0^2 + C_0x_0^2y_0 + \cdots\cdots + D_2x_0 + D_3$$

a condition linear in the quantities $(A, B_0, B_1, C_0, \ldots, D_2, D_3)$. Some one of these being fixed as unity, to determine the other nine will require at least 9 linear equations,—9 if all are linearly independent. Hence 8 are never sufficient; through 8 points can pass at least two independent cubics. The entire result of this paragraph is therefore the proof of the following theorem.

THEOREM. *A cubic curve may be passed through any nine points of a plane. If no four of the nine points are in any line, no seven are on any conic, and no six on a conic when the three others are on a line, the nine will in general determine completely a proper cubic. But to any eight points a ninth can be found, such that the cubic through the nine points is indeterminate. This ninth point is itself indeterminate when four of the eight lie on a line, or seven of the eight lie on a conic; in both these cases the cubic must be degenerate, and any ninth point not on the line in the one case, not on the conic in the other case, will complete its determination.* For convenience these theorems are usually but inaccurately condensed into the two formulae: *Nine points determine a cubic;* and *Eight points on a cubic determine a ninth; and all cubics containing the eight contain also the ninth.*

To finish the demonstration of the first part, pass two cubics: $\varphi = 0$ and $\psi = 0$, through any 8 of the given 9 points, $A_1, A_2, \ldots, A_8$. In A_9 let φ and ψ have values φ_9 and ψ_9, not both equal to zero. Determine λ' by the condition:

$$\varphi_9 + \lambda' \psi_9 = 0.$$

Since every cubic through the first 8 points has an equation of the form $\varphi + \lambda\psi = 0$, and since only one value of λ makes $\varphi + \lambda\psi$ equal to zero in the ninth point, the curve is completely determined. It can be degenerate only in the three excluded cases. The latter part of the theorem is demonstrated above, in the discussion of systems of 8 points.

§ 4. Examples of nine points on two cubics.

If any cubic curve, proper or degenerate, cuts out six points on a conic and three on a line, these nine form evidently a

set which does not completely determine a cubic; if another cubic, proper or degenerate, contain eight of such points it must contain the other also. Proof will be given in the next paragraph. This principle furnishes a ready proof of certain theorems on conics.

THEOREM: *Three conics having two points in common intersect two and two in three other pairs of points, and the three lines joining those pairs of points pass through a common point.*

PROOF. Denote by A_1, A_2, the fundamental points of the three conics S_1, S_2, S_3, (Fig. 8), and their other intersections by B_1, B_2,; C_1, C_2; D_1, D_2. Call the intersection of the lines $B_1\ B_2$ and $C_1\ C_2$, O; then the line $D_1\ D_2$ must pass through O. For the first 8 points lie on three degenerate cubics, viz. upon the conic S_1 and the line $B_1\ B_2$, the conic S_2 and the line $C_1\ C_2$, the conic S_3 and the line $D_1\ D_2$.

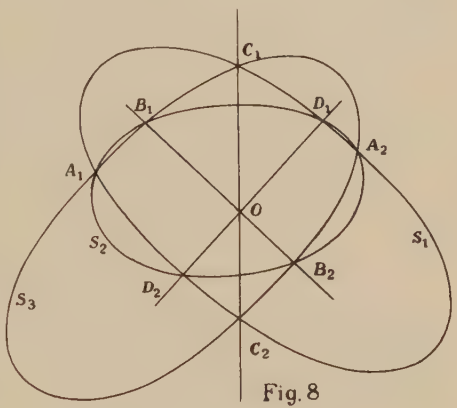


Fig. 8

The point O lies as ninth point upon the first two, therefore also upon the third. But S_3 meets the first cubic in six points, $A_1\ A_2\ C_1\ C_2\ B_1\ B_2$, hence the remaining intersections must fall upon the other branch; that is, the line $D_1\ D_2$ must contain O. q. e. d.

THEOREM (Pascal's). If six points on a conic be joined in pairs by two distinct sets of three lines, the remaining intersections of lines of one set with lines of the other set are three points on a right line.

PROOF. This is a particular case of the preceding theorem, the conics S_1 and S_3 becoming pairs of lines (Fig. 9). Hence the proof is the same.

THEOREM. If three conics S_1, S_2, S_3 have three points in

common, and intersect further in three points: S_2 and S_3 in P_1, S_3 and S_1 in P_2, S_1 and S_2 in P_3, then through P_1, P_2 and P_3 will pass the sides of an infinite number of triangles whose opposite vertices lie respectively upon S_1, S_2 and S_3.

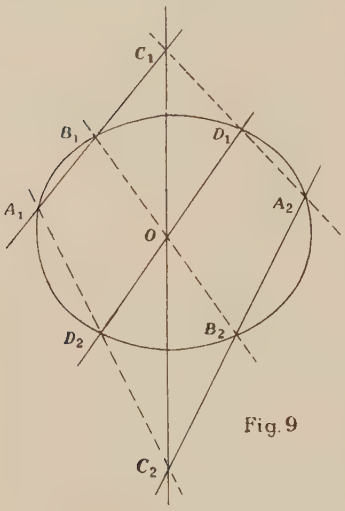


Fig. 9

PROOF. Construct a diagram (Fig. 10) as described. Through P_1 pass a line l_1 in any arbitrary direction, meeting S_2, S_3 in A_2, A_3 respectively. Join P_2, A_3 by a line l_2 meeting S_1 in A_1. Denote by l_3 the line $P_3\,A_2$. There are now three degenerate cubics which contain eight points in common; the three fundamental points, and P_1, P_2, P_3, A_2, A_3. These are:

1) the conic S_1 and the line l_1,
2) the conic S_2 and the line l_2,
3) the conic S_3 and the line l_3.

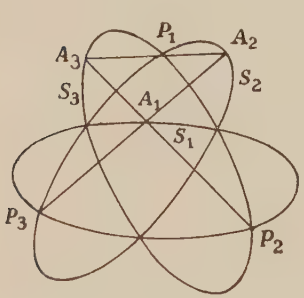


Fig. 10

The first two intersect in a ninth point A_1, hence A_1 must lie on S_3 or l_3; but it lies on S_1 and generally not in one of the fundamental points, so that it cannot lie upon S_3, therefore it is on l_3. The triangle $A_1\,A_2\,A_3$ is such a triangle as the theorem describes, and A_3 is any point whatever, four excepted, upon S_3, therefore the number of such triangles is infinite. q. e. d.

§ 5. Satellite lines. Two triads of secants meeting in nine points on a cubic.

The following is a most important corollary of the theorem on the derivative ninth point.

THEOREM. *Two cubics intersect in* 9 *points: if* 6 *of these*

lie on a conic, the remaining 3 *lie on a line*; *and if* 3 *lie on a line, the remaining* 6 *are on a conic.*

PROOF. In the first case the conic and the line determined by two of the remaining points constitute a cubic through 8 of the points in question; therefore the ninth, which cannot lie on the conic (II, § 1), must lie on the line. In the second case the same argument is made for the line through the 3 points and the conic through 5 of the remaining points.

Let one of the cubics be a proper cubic; while the second shall be degenerate, consisting of three lines, and the conic also shall consist of two lines. The resulting theorem may be stated thus:

THEOREM. *If three lines join in pairs the six intersections of two secants with any cubic, they meet the cubic in three additional points, and these derived points lie on a right line* (Fig. 11).

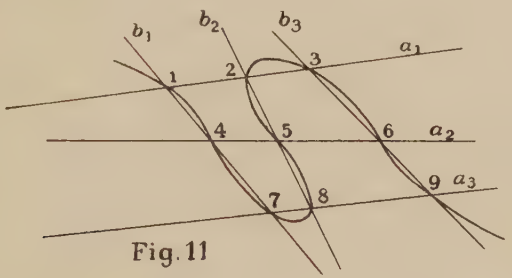


Fig. 11

When the two secants are distinct, their points may be connected in pairs arbitrarily, so that the closing line may be any one of six; but as a limiting case the two secants may coincide, then the only joining lines that remain distinct from the double secant are those which become tangents to the cubic at the points of intersection. Hence a theorem containing a definition.

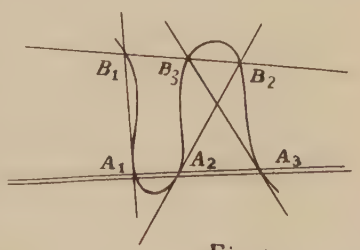


Fig. 12

THEOREM. *Tangents drawn to a cubic at three collinear points meet the curve again in three points which are collinear.*

DEFINITION. *The line containing the three tangential points is called the satellite of the line on which lie the three points of contact.**

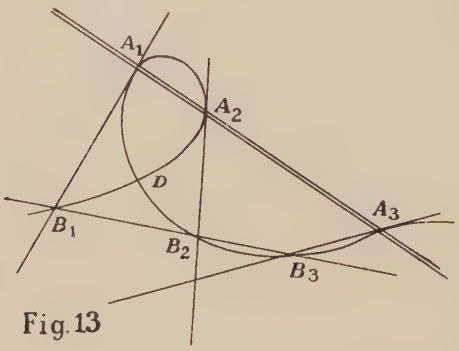


Fig. 13

This theorem is illustrated in Figures 12 and 13, where the line $B_1\ B_2\ B_3$ is the satellite of the line $A_1\ A_2\ A_3$. Any line through D, Fig. 13, will have its satellite indeterminate, but for such cases the satellite can be defined as a limit, so that the satellite of any line through D will be another line through D: of each tangent on D, for example, the other tangent.

§ 6. Lines associated with a conic and a cubic.

Instead of the two secants of Fig. 11 one may take any conic with its six points of intersection upon the cubic. The argument is correspondingly modified, and the conclusion is this:

THEOREM. *The six intersections of a conic with a proper cubic may be joined in pairs in any order by three secants: these cut the curve again in three points which lie in a straight line.*

If possible, let the intersections of the conic and the cubic be three pairs of coincident points, i. e. let the conic touch the cubic in three points (III, § 14). Then this theorem is seen to include as limiting cases the following three.

THEOREMS. *Tangents drawn at three points where a cubic is touched by a conic meet the cubic in three collinear points.*

* Theorem cited by DURÈGE from MACLAURIN: De linearum geometricarum proprietatibus. Term "Satellite" from CAYLEY, Phil. Trans., Vol. 147, p. 416. See for discussion on satellite of line at infinity PLUECKER: System der Analytischen Geometrie (1835), p. 132 seqq.

The three points of contact between a cubic and a conic are vertices of a triangle whose sides meet the cubic again in three collinear points.

If a chord be drawn joining any two points of contact of a conic and a triply tangent cubic, the tangent at its third intersection with the cubic and the tangent at the third contact of the conic and cubic meet the cubic in a common point. Hence if two points of contact are taken arbitrarily, the number of conics touching the cubic in those two points and once elsewhere cannot be greater than the number of tangents to the cubic from one of its points, less one. (See III, § 3.)

§ 7. A line coincident with its satellite meets the cubic in three points of inflexion.

The foregoing and their less simple converse theorems are often of use; more interesting however is the case where the two lines of Fig. 12 come to coincide in a line joining two points of inflexion. If a line touches a cubic in A and meets it again in B, B is called the *tangential point* of A (§ 5). The satellite b of a line a contains the tangential points of the three intersections of a with the cubic. If now a line be drawn through two points of inflexion, its satellite must coincide with it; for the tangential points of the two inflexions coincide with those inflexions. But a meets the curve in a third point, and b must meet it in the same point, which thus coincides with its tangential and is therefore a point of inflexion. Therefore: *Every line containing two points of inflexion contains also a third.* If the first two are real, their join line is real, and also its third intersection with the curve. Since a cubic has three inflexions on an odd branch (I, § 5), *there must be at least one real line containing three real inflexions of any cubic which has no singular point.*

Pushing the inquiry further, we should see that if there is a fourth real inflexion, there must be at least five more, or altogether not less than nine, connected by not fewer than twelve real lines. Later we shall prove that the exact number of real inflexions is three (IV, § 4).

§ 8. Special forms of the equation of a cubic.

From the foregoing theorems can be derived certain forms into which the equation of a cubic can be thrown. If two arbitrary lines: $p = 0$ and $q = 0$, cut the curve, and three other lines: $t = 0$, $u = 0$, $v = 0$, join in pairs those intersections, these three meet the cubic again in the points of another line: $r = 0$ (II, § 6). All 9 points lie on two independent degenerate cubics, namely on $pqr = 0$ and on $tuv = 0$. The equation of any third cubic through the nine points must be of the form

$$F = 0 = \varkappa \cdot pqr + \lambda \cdot tuv, \tag{1}$$

by (II, § 3), therefore for some value of $\varkappa : \lambda$ this is the equation of the fundamental cubic. Every proper cubic can have its equation brought into the form (1), and of the linear expressions p, q, r, two are perfectly arbitrary: when two are chosen, the other linear expressions can be determined in six ways, for there are six ways of connecting in pairs three points on one line with three points on another.

If $S = 0$ denote an arbitrary conic, $t = 0$, $u = 0$, $v = 0$, three chords through three pairs of its intersections with the cubic: $F = 0$, then some line: $p = 0$ will contain the remaining intersections of t, u, v, with the curve. Two independent equations are here

$$S \cdot p = 0 \quad \text{and} \quad t \cdot u \cdot v = 0,$$

therefore as above we shall find for some value of $\lambda : \varkappa$,

$$F \equiv \varkappa \cdot Sp + \lambda \cdot tuv = 0 \tag{2}$$

as the equation of the fundamental curve. Every proper cubic can have its equation put into this form, and either the quadric S or two of the three linear expressions: t, u, v may be chosen arbitrarily. If S be taken as the square of a linear form: $S = s^2$, the equation becomes:

$$F \equiv \varkappa \cdot s^2 p + \lambda \cdot tuv = 0, \tag{3}$$

and $t = 0$, $u = 0$, $v = 0$ are lines tangent to the curve upon the line: $s = 0$, the satellite being the line: $p = 0$. The equation of every proper cubic can be brought into the form (3); either the line: $s = 0$ may be assumed arbitrarily, or any two tangents as $t = 0$ and $u = 0$. It will be proved later that the satellite: $p = 0$, might be chosen at random.

As a special case of equation (3), let the line $s = 0$ recede to infinity:* in Cartesian coordinates the equation is then:

$$F(x, y, 1) \equiv \varkappa \cdot p + \lambda \cdot tuv = 0. \tag{4}$$

If finally the line through three inflexions be taken as $p = 0$, and t, u, v the inflexional tangents, we have the form:†

$$F \equiv \varkappa \cdot p^3 + \lambda \cdot tuv = 0. \tag{5}$$

Every non-singular real cubic equation can be put under the form (5), in at least one way, where $p = 0$ denotes a real line through 3 inflexions, and $t = 0$, $u = 0$, $v = 0$ denote tangents in those points of inflexion (I, § 5).

§ 9. On normal forms and the counting of constants.

Any specified possible form of an equation or system of equations, useful or convenient for the solution of a particular problem, may be called a normal form. Thus for the conic, $x_1^2 + x_2^2 + x_3^2 = 0$ is a normal form, convenient when the relations of self-conjugate triangles are to be examined. Oftener however the term: normal form, is used when the proposed form is unique or when it can be produced by the admissible transformations in only a finite number of ways. When orthogonal transformations of rectangular coordinates (x, y) are admitted, and no others, then for central conics

$$\frac{x^2}{a} + \frac{y^2}{b} = 1$$

* See construction depending on this equation: PLUECKER, System der Analytischen Geometrie, p. 166.

† PLUECKER, l. c., p. 282.

is a unique normal form. For two conics whose relative position is not special, among all possible choices of triangle of reference, only one reduces the equations simultaneously to the forms:

$$x_1^2 + x_2^2 + x_3^2 = 0 \quad \text{and} \quad a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 0.$$

This is therefore a unique normal form for the system of two quadrics when arbitrary linear transformations of homogeneous coordinates are admitted.

As a proof that the latter is a possible form, one may first count the arbitrary constants involved and compare the number of conditions to be satisfied. The equations: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ of the proposed lines of reference contain implicitly 9 constants, the quantities a_1, a_2, a_3 are 3, so that we have altogether 12, exactly the number of constants in the two ternary quadrics together. But further it would be necessary to show that the equations of condition would admit at least one solution; that is, that one could not eliminate simultaneously all the arbitrary constants without eliminating also all the constants of the given quadric equations. If all the proposed arbitrary constants could be eliminated from the equations of condition, leaving a resultant not identically zero, that would show that the proposed form is admissible only for a special class of cases, for those namely in which the resultant, as a function of the coefficients of the given equations, is equal to zero. For example, the equation of a conic can take the form: $x_1 x_2 = 0$, only when its discriminant vanishes.

If any solution of the equations of condition is possible, then by counting the exact number of solutions geometric theorems may be found. To repeat: *no form can be established algebraically as a normal form until it is shown that the equations of condition to be solved do not imply any algebraic equation among the given constants.* Usually this proof amounts to the actual production of the resultant from the equations of condition, and the verification of the hypothesis that it is identically equal to zero.

This proof may be often shortened by the use of previously known geometrical theorems. For the cubic, the normal form

$$F = \varkappa \cdot p^3 + \lambda \cdot tuv = 0$$

is a case in point. Counting the constants we find in the forms:

$$\begin{aligned} p &= x_1 + p_2 x_2 + p_3 x_3, \\ t &= x_1 + t_2 x_2 + t_3 x_3, \\ u &= x_1 + u_2 x_2 + u_3 x_3, \\ v &= x_1 + v_2 x_2 + v_3 x_3, \end{aligned}$$

together with $\varkappa$, λ, 10 constants, which is exactly the number of coefficients in a general cubic. But the equations of condition are of the third degree in the quantities $p_2, p_3, t_2, \ldots, v_3$, so that systematic elimination would be very laborious. This is rendered unnecessary by the geometrical theorem (I, § 5) that at least one inflexional line ($p = 0$) exists, together with real tangents $t = 0$, $u = 0$, $v = 0$, at its inflexions. That this form is unique is however not proven; and algebraic analysis will show (IV, § 2) that other inflexional lines exist, whose corresponding inflexional tangents however are imaginary.

Provisionally can be admitted, as containing the required number of constants, the two other important normal forms of the cubic:

$$0 = F \equiv p^3 + q^3 + r^3 + c \cdot pqr, \qquad \text{(Hesse)},$$

and

$$0 = F \equiv 4p^3 - q^2 r - g_2 \cdot pr^2 - g_3 r^3, \qquad \text{(Weierstrass)},$$

where g_2, g_3 and c are constants.

Chapter III.

Poles and polars with respect to a cubic. The Hessian, the Cayleyan, and the poloconics.

§ 1. Certain Notations and Abbreviations.

Having constantly occasion to write equations, and because the number of terms in a cubic and its related equations makes it inconvenient to write them in full, we use abbreviations. These are not very intricate, and shorten our calculations. When coefficients are to be written explicitly we shall use subscript indices to designate their place in the equation, thus:

$$\begin{aligned} f \text{ or } f(x) \text{ or } f(x_1, x_2, x_3) \equiv\ & a_{300}x_1^3 + a_{030}x_2^3 + a_{003}x_3^3 \\ & + 3\,a_{210}x_1^2x_2 + 3\,a_{021}x_2^2x_3 + 3\,a_{102}x_3^2x_1 \\ & + 3\,a_{120}x_1x_2^2 + 3\,a_{012}x_2x_3^2 + 3\,a_{201}x_3x_1^2 \\ & + 6\,a_{111}x_1x_2x_3 \\ \equiv\ & \sum_{\alpha+\beta+\gamma=3} \frac{3!}{\alpha!\,\beta!\,\gamma!}\, a_{\alpha\beta\gamma}x_1^\alpha x_2^\beta x_3^\gamma. \end{aligned}$$

More briefly we shall sometimes write

$$(ax)^3 \text{ instead of } f(x_1, x_2, x_3) \text{ or } \sum_{\alpha+\beta+\gamma=3} \frac{3!}{\alpha!\,\beta!\,\gamma!}\, a_{\alpha\beta\gamma}x_1^\alpha x_2^\beta x_3^\gamma.$$

and so of course

$$(ay)^3 \text{ instead of } f(y_1, y_2, y_3),$$

and similarly

$$(a, x + \lambda y)^3 \text{ instead of } f(x_1 + \lambda y_1, x_2 + \lambda y_2, x_3 + \lambda y_3).$$

These are merely abbreviations, but the reasons of their convenience will soon appear. For the equation of a line we can use indifferently:

$$u = 0, \quad (ux) = 0, \quad u_1x_1 + u_2x_2 + u_3x_3 = 0, \quad \sum u_ix_i = 0,$$

the third being the full explicit form. The equation of a conic can be indicated by:

$$0 = F = F(x) = F(x_1, x_2, x_3) = (Ax)^2 = (A_1x_1 + A_2x_2 + A_3x_3)^2$$

$$= A_{200}x_1^2 + 2A_{110}x_1x_2 + \text{etc.} = \sum_{\alpha+\beta+\gamma=2} \frac{2!}{\alpha!\,\beta!\,\gamma!} A_{\alpha\beta\gamma}x_1^\alpha x_2^\beta x_3^\gamma.$$

Of these the forms $F(x)$ and $(Ax)^2$, as shorter, are naturally more often used than the others.

When partial differential coefficients occur, as in equations of tangents, the notation becomes slightly more complicated. Symbols like f_1, f_2, f_3, F_1, F_{12}, f_{123} are common in textbooks on differential calculus. The last one of these we shall not use in its customary sense, but replace it by f_{111}, in accordance with the notation already explained for cubics. To define these exactly: $f_1(x) = \frac{\partial f(x)}{\partial x_1}$, $f_2(x) = \frac{\partial f(x)}{\partial x_2}$, $f_3(x) = \frac{\partial f(x)}{\partial x_3}$;

$$F_1(x) = \frac{\partial F(x)}{\partial x_1}, \quad F_{12}(x) = \frac{\partial^2 F(x)}{\partial x_1\,\partial x_2}, \quad f_{111}(x) = \frac{\partial^3 f(x)}{\partial x_1\,\partial x_2\,\partial x_3};$$

$$f_{112}(x) = \frac{\partial^4 f(x)}{\partial x_1\,\partial x_2\,\partial x_3^2}, \text{ etc.}$$

It is obvious that we can distinguish $f_{12}(x)$ from $f_{12}(y)$, thus

$$f_{12}(x) = \frac{\partial^2 f(x)}{\partial x_1\,\partial x_2}, \quad f_{12}(y) = \frac{\partial^2 f(y)}{\partial y_1\,\partial y_2} = [f_{12}(x)]_{(x)\ (y)}.$$

More often however it will be convenient to write differential coefficients as follows. If $f(x)$ denotes the same thing as $(ax)^3$, then $f_1(x)$ will denote the same as $3a_1(ax)^2$. By adopting this convention we can apply the ordinary rules of differentiation to our abbreviated symbols; if $(ax)^3 = (a_1x_1 + a_2x_2 + a_3x_3)^3$ denoted the cube of an explicit linear function $a_1x_1 + a_2x_2 + a_3x_3$, we should of course have $\frac{\partial}{\partial x_1}(ax)^3 = 3a_1(ax)^2$. Indeed that

is the reason for using $3a_1(ax)^2$ as a sign for $\frac{\partial}{\partial x_1}(ax)^3$. To save the labor of restoring the entire form $(ax)^3$ and differentiating, it is only necessary to notice that the explicit notation results from the abbreviation simply by writing instead of $a_1^\alpha a_2^\beta a_3^\gamma$ the coefficient $a_{\alpha\beta\gamma}$. In short, then, we shall use these signs interchangeably:

$$a_1^\alpha a_2^\beta a_3^\gamma = a_{\alpha\beta\gamma}$$

$$= \frac{1}{(\alpha+\beta+\gamma)!} \frac{\partial^{\alpha+\beta+\gamma}}{\partial x_1^\alpha \partial x_2^\beta \partial x_3^\gamma}(ax)^3; \quad (\alpha+\beta+\gamma = 3).$$

So the abbreviations: $a_1a_2(ax)$ or $a_{12}(ax)$ will denote

$$\frac{1}{3\cdot 2}\frac{\partial^2(ax)^3}{\partial x_1 \partial x_2} \text{ and also } \frac{1}{2}\frac{\partial}{\partial x_2}[a_1(ax)^2];$$

we shall write also similarly: $a_1a_1(ax)$ or $a_1^2(ax)$ for $\frac{1}{3\cdot 2}\frac{\partial^2}{\partial x_1^2}(ax)^3$ or $\frac{1}{6}f_{11}(x)$.

In these short notations, the equation of the polar of a point (y), i.e. (y_1, y_2, y_3) with respect to a conic

$$(Ax)^2 = F(x) = 0$$

can be written in the second or third of the following forms, instead of the first:

$$y_1\frac{\partial F(x)}{\partial x_1} + y_2\frac{\partial F(x)}{\partial x_2} + y_3\frac{\partial F(x)}{\partial x_3} = 0,$$

$$y_1F_1(x) + y_2F_2(x) + y_3F_3(x) = 0,$$

$$2y_1A_1(Ax) + 2y_2A_2(Ax) + 2y_3A_3(Ax) = 0.$$

This third form can be further condensed into:

$$2(A_1y_1 + A_2y_2 + A_3y_3)(Ax) = 0.$$

or simplest of all, —

$$2(Ay)(Ax) = 2(Ax)(Ay) = 0.$$

Occasionally slight variations from these precise notations will be convenient.

§ 2. Definition of Polars. Projection transforms Polars into Polars.

In the theory of conics a polar is usually defined by some geometric property, and from this its equation is deduced. For the cubic and for curves of higher order it is simpler to define polars algebraically by the form of their equations, and afterward to examine their geometric properties. This order can be regarded as the first step in a general plan in projective geometry: *to investigate all curves whose relation to the fundamental curve is not altered by projection.* The algebraic equivalent of any projection is some linear transformation of the variables which represent coordinates. After defining polars we shall therefore prove that any polar of a given cubic is transformed by linear substitution of the variables into the corresponding polar of the transformed cubic.

Definition. The first polar of a point (y) with respect to a cubic: $f(x) \equiv (ax)^3 = 0$, is the curve whose equation results from one application of the so-called polar operator $\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_3}\right)$ or briefly $\left(y \frac{\partial}{\partial x}\right)$ to the equation of the cubic: $(ax)^3 = 0$. In abbreviated notation its equation is:

$$y_1 \frac{\partial f(x)}{\partial x_1} + y_2 \frac{\partial f(x)}{\partial x_2} + y_3 \frac{\partial f(x)}{\partial x_3}$$
$$\equiv 3y_1a_1(ax)^2 + 3y_2a_2(ax)^2 + 3y_3a_3(ax)^2$$
$$\equiv 3(ay)(ax)^2 = 0.$$

Definition. The second polar of the point (y) with respect to the cubic: $f(x) \equiv (ax)^3 = 0$ is the curve whose equation results from two consecutive applications of the polar operator:

$\left(y\frac{\partial}{\partial x}\right)$, viz. $\left(y\frac{\partial}{\partial x}\right)^2$, to the equation of the cubic; the same equation will result from one application of the polar operator $\left(y\frac{\partial}{\partial x}\right)$ to the equation of the first polar of (y) with respect to the cubic. In abbreviated notation the equation becomes:—

$$\left(y\frac{\partial}{\partial x}\right)^2 f(x) \equiv \left(y\frac{\partial}{\partial x}\right)\left[\left(y\frac{\partial}{\partial x}\right)f(x)\right] \equiv \left(y\frac{\partial}{\partial x}\right)[3(ay)(ax)^2]$$

$$\equiv 3(ay)\left[\left(y\frac{\partial}{\partial x}\right)(ax)^2\right] = 6(ay)(ay)(ax)$$

$$- 6(ay)^2(ax) = 0.$$

For any real conic it was intuitively evident that the geometric relation of pole, curve and polar was not to be altered by projection; for tangents to the curve from the pole touch it where the polar intersects it, and projection transforms tangents to the old curve into tangents to the new curve, so that the transformed diagram is essentially like the former. A similar geometric form of argument is possible, not very complicated for the cubic, but more so in increasing degree for curves of higher orders. In the algebraic form of proof, on the other hand, we attend only to the polar operator $\left(y\frac{\partial}{\partial x}\right)$, regardless of the order of the curve, so that one demonstration suffices for first polars and second polars of a cubic, and for polars of all orders with respect to curves of whatever degree. The theorem is as follows.

THEOREM. *The polar operator* $\left(y\frac{\partial}{\partial x}\right)$ *is invariable under transformation by linear substitution. That is to say, if the variables* y_1, y_2, y_3 *and* x_1, x_2, x_3 *are expressed by the same linear substitution in terms of* y_1', y_2', y_3' *and* x_1', x_2', x_3' *respectively, then the polar operator* $\left(y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + y_3\frac{\partial}{\partial x_3}\right)$ *is identically equal to the polar operator in transformed variables:* $\left(y_1'\frac{\partial}{\partial x_1'} + y_2'\frac{\partial}{\partial x_2'} + y_3'\frac{\partial}{\partial x_3'}\right)$.

PROOF. Let the linear substitutions be the following, the α_i, β_i and γ_i denoting perfectly arbitrary constants or parameters.

$$(1)\qquad \left.\begin{aligned} x_1 &= \alpha_1 x_1' + \beta_1 x_2' + \gamma_1 x_3' \\ x_2 &= \alpha_2 x_1' + \beta_2 x_2' + \gamma_2 x_3' \\ x_3 &= \alpha_3 x_1' + \beta_3 x_2' + \gamma_3 x_3' \end{aligned}\right\}, \quad \left.\begin{aligned} y_1 &= \alpha_1 y_1' + \beta_1 y_2' + \gamma_1 y_3' \\ y_2 &= \alpha_2 y_1' + \beta_2 y_2' + \gamma_2 y_3' \\ y_3 &= \alpha_3 y_1' + \beta_3 y_2' + \gamma_3 y_3' \end{aligned}\right\}$$

Call the determinant of the substitution Δ, its first minors A_i, B_i, C_i.

$$\Delta = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

$$A_i = \frac{\partial \Delta}{\partial \alpha_i}, \quad B_i = \frac{\partial \Delta}{\partial \beta_i}, \quad C_i = \frac{\partial \Delta}{\partial \gamma_i}, \qquad (i = 1, 2, 3).$$

Solving the substitution equations (1) for y_1', y_2', y_3', we have

$$(3)\qquad \begin{aligned} \Delta \cdot y_1' &= A_1 y_1 + A_2 y_2 + A_3 y_3 \\ \Delta \cdot y_2' &= B_1 y_1 + B_2 y_2 + B_3 y_3 \\ \Delta \cdot y_3' &= C_1 y_1 + C_2 y_2 + C_3 y_3 \end{aligned}$$

Partial differentiation is to be applied to functions of x_1, x_2, x_3, alone; hence we have the formal identities:

$$\frac{\partial}{\partial x_1'} = \frac{\partial x_1}{\partial x_1'} \cdot \frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial x_1'} \cdot \frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial x_1'} \cdot \frac{\partial}{\partial x_3}.$$

Now from the substitution equations we find

$$\frac{\partial x_1}{\partial x_1'} = \alpha_1, \quad \frac{\partial x_2}{\partial x_1'} = \alpha_2, \quad \frac{\partial x_3}{\partial x_1'} = \alpha_3, \text{ etc., etc.}$$

Therefore:

$$(4)\qquad \begin{aligned} \frac{\partial}{\partial x_1'} &= \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}. \\ \frac{\partial}{\partial x_2'} &= \beta_1 \frac{\partial}{\partial x_1} + \beta_2 \frac{\partial}{\partial x_2} + \beta_3 \frac{\partial}{\partial x_3}. \\ \frac{\partial}{\partial x_3'} &= \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} + \gamma_3 \frac{\partial}{\partial x_3}. \end{aligned}$$

Multiply these three expressions by those given above for y_1', y_2', y_3' respectively, and add.

$$\Delta\left(y_1'\frac{\partial}{\partial x_1'}+y_2'\frac{\partial}{\partial x_2'}+y_3'\frac{\partial}{\partial x_3'}\right) \equiv (\alpha_1A_1+\beta_1B_1+\gamma_1C_1)y\,\frac{\partial}{\partial x_1}$$
$$+(\alpha_2A_2+\beta_2B_2+\gamma_2C_2)y_2\frac{\partial}{\partial x_2} + (\alpha_3A_3+\beta_3B_3+\gamma_3C_3)y_3\frac{\partial}{\partial x_3}$$
$$\equiv \Delta\left(y_1\frac{\partial}{\partial x_1}+y_2\frac{\partial}{\partial x_2}+y_3\frac{\partial}{\partial x_3}\right)$$

since $\alpha_1A_1+\alpha_2A_2+\alpha_3A_3 \equiv \Delta$, etc. Removing the common factor Δ, we have the identity asserted in the theorem:

$$(5) \qquad y_1'\frac{\partial}{\partial x_1'}+y_2'\frac{\partial}{\partial x_2'}+y_3'\frac{\partial}{\partial x_3'} \equiv y_1\frac{\partial}{\partial x_1}+y_2\frac{\partial}{\partial x_2}+y_3\frac{\partial}{\partial x_3}\,.$$

Denote the result of substituting in $(ax)^3$ the values of x_1, x_2, x_3, from (1) by $(a'x')^3$; in other words, let (ax) and $(a'x')$ be identically equal by virtue of the substitution equations (1). The identity between the two polar operators then gives identity between two first polars and two second polars by virtue of equations (1), viz.

$$\left(y'\frac{\partial}{\partial x'}\right)(a'x')^3 \equiv \left(y\frac{\partial}{\partial x}\right)(ax)^3 \text{ or } (a'y')(a'x')^2 \equiv (ay)(ax)^2\,,$$
$$\left(y'\frac{\partial}{\partial x'}\right)^2(a'x')^3 \equiv \left(y\frac{\partial}{\partial x}\right)^2(ax)^3 \text{ or } (a'y')^2(a'x') \equiv (ay)^2(ax)\,.$$

For curves of higher order the theorem is true for third polars, fourth polars, in fact for polars of every order.

§ 3. Geometric Properties of Polars. Class of curves of order three.

By definition of polars, every point (y) in the plane of a cubic curve has associated with it a first or conic polar and a second or linear polar. It will be found directly that every line of the plane has one or more points as poles, a line being determined by the same number of constants as a point. But this will not be true of all conics in the plane; there is a five-fold infinity of conics, and only a two-fold infinity of points, hence only a two-fold infinity out of the existing five-fold infinity of conics can occur as polars of points with respect

to a given cubic. The question arises therefore, if a real cubic were drawn, and a real conic intersecting it in six real points, how can one decide whether the conic does or does not belong to the system of conic polars with respect to the cubic? This is answered by the following theorem.

THEOREM. *The polar conic of a point meets the cubic in the points of contact of tangents drawn from the pole to the cubic; and conversely, tangents to the cubic at its intersections with any polar conic pass through a common point, the pole of that conic. This gives a complete criterion.*

PROOF. Let (y) be the pole, (x) a variable point on the cubic, so that $(ax)^3 = 0$. On a line joining the points (x) and (y) the coordinates of an indeterminate point are: $z_i = x_i + \lambda y_i$ $(i = 1, 2, 3)$. Let the point (z) lie also on the curve, and develope in powers of λ the equation: $(az)^3 = (a, x + \lambda y)^3 = 0$.

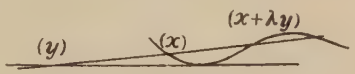


Fig. 14

$$(a, x + \lambda y)^3 = (ax)^3 + 3\lambda (ax)^2 (ay) + 3\lambda^2 (ax)(ay)^2 + \lambda^3 (ay)^3 = (az)^3 = 0.$$

Since $(ax)^3 = 0$, one term disappears, and one root of this equation is already $\lambda_1 = 0$. To make the line $\overline{xy}$ tangent in (x) to the curve, a second root λ_2 must become zero. This occurs only if the coefficient of λ vanishes; we must have

$$(ax)^2 (ay) = 0.$$

There are thus two equations satisfied by a point of contact (x) of any tangent from (y). The first, $(ax)^3 = 0$, requires (x) to lie on the cubic. The second, $(ay)(ax)^2 = 0$, requires (x) to lie on the conic polar of (y), the assertion of the theorem. Conversely these two conditions are sufficient to make two roots, λ_1 and λ_2, equal to zero, hence the six intersections of cubic and conic polar are all points of contact of tangents drawn from the pole, i. e. these six tangents all contain the pole as a common point.

The number of tangents that can be drawn from an arbitrary point to any algebraic curve is called the *class* of that curve.

Accordingly *the curve of the third order is generally of the sixth class*: exceptions must be made if conic polars are found to meet the cubic in any points not proper contacts, as in a double point or cusp. How many tangents then can be drawn to a general cubic from a point on the curve? As above, we have for the three intersections of a line $\overline{xy}$ with the curve the equation of third degree in λ:

$$0 = 3\lambda(ax)^2(ay) + 3\lambda^2(ax)(ay)^2 + \lambda^3(ay)^3,$$

since (x) is taken as a point on the curve. Remove the factor λ, because the root $\lambda = 0$ gives the point (x) itself.

$$0 = 3(ax)^2(ay) + 3\lambda(ax)(ay)^2 + \lambda^2(ay)^3.$$

For tangency, the remaining two roots must be equal; the discriminant of the quadric in λ must vanish.

$$D = 9[(ax)(ay)^2]\cdot[(ax)(ay)^2] - 12[(ax)^2(ay)]\cdot[(ay)^3] = 0.$$

This is of degree 4 in (y). This point (y) is otherwise unrestricted, and may be assumed to lie on any particular straight line. That line meets the curve: $D = 0$, in four points and each of these points lies on a tangent from the point (x), therefore *there are four tangents from* (x) *to other points of the cubic*. This equation $D = 0$ can be used in determining the cross-ratios of the four tangents (V, § 4).

Returning to the equation of third degree in λ, observe that if $(ax)^3 = 0$ and $(ax)^2(ay) = 0$, the line $\overline{xy}$ touches the cubic in (x). But it may be not a determinate line for some special point (x). If $(ax)^3 = 0$, and the equation $(ax)^2(ay) = 0$ is satisfied for all points (y) of the plane, then every line through (x) would meet the curve there in two coincident points; that is, (x) must be a double point or cusp. This is apparent intuitively if (x) is a real point on a real branch, but to cover imaginary points and isolated double points it is necessary to fix algebraically a definition of multiple points of a curve.

Definition. If all conic polars with respect to a cubic have a common point, that is, if for some point (x) the

equation $(ax)^2(ay) = 0$, is satisfied for all points (y) in the plane, the point (x) is called a multiple point. This implies the three conditions: $(ax)^2 a_1 = (ax)^2 a_2 = (ax)^2 a_3 = 0$; and since (x) is itself a point (y) of the plane, it involves: $(ax)^2(ax) = (ax)^3 = 0$, i. e. the point (x) is necessarily a point of the cubic.

If the cubic f has a double point or cusp, the conic polar φ of a point (y) has at least two intersections at the multiple point, so that the class of the curve is reduced by at least two units; for each branch of the curve in the multiple point intersects the conic on one point. If the conic were tangent to either branch, there would be three or more intersections in the double point. That this is not the case, we can verify upon one example, proving this theorem:

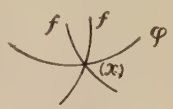


Fig. 15

THEOREM. *The conic polar of an arbitrary point with respect to a cubic having an ordinary double point is not tangent to either branch of the cubic in that point. The number of coincident points of intersection is exactly two, and the cubic having an ordinary double point is therefore of class* 4.

PROOF. Represent the cubic by the equation:

$$f(x) = (ax)^3 = x_1x_2x_3 + ax_1^3 + bx_1^2x_2 + cx_1x_2^2 + dx_2^3 = 0.$$

The tangents in the double point: $x_1 = x_2 = 0$, are the lines:

$$x_1 \quad 0. \qquad x_2 \quad 0.$$

The polar of the arbitrary point (1, 1, 0) has the equation:

$$(ay)(ax)^2 = 1\,(x_2x_3 + 3\,ax_1^2 + 2\,bx_1x_2 + cx_2^2) + 1\,(x_1x_3 + bx_1^2 + 2\,cx_1x_2 + 3\,dx_2^2) = 0.$$

At the double point: $x_1 = x_2 = 0$, the tangent to this conic polar is the line: $x_1 + x_2 = 0$, a line different from either tangent to the cubic. q. e. d.

To investigate the class of a cubic with a cusp, take an equation of most general form for such a curve:

$$f(x) = x_1^2x_3 + ax_1^3 + bx_1^2x_2 + cx_1x_2^2 + dx_2^3 = 0.$$

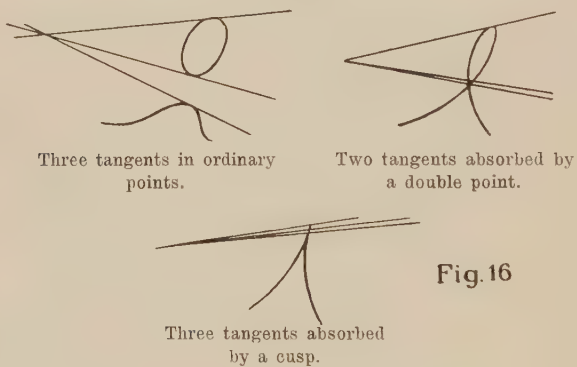
The conic polar of (y) with respect to this is given by:

$$\varphi(x) = y_1(2x_1x_3 + 3ax_1^2 + 2bx_1x_2 + cx_2^2) + y_2(bx_1^2 + 2cx_1x_2 + 3dx_2^2) + y_3x_1^2 = 0.$$

This conic has in (0, 0, 1) the tangent:

$$2y_1x_1 = 0, \text{ or } x_1 = 0,$$

which is the cuspidal tangent of the cubic in that point. This tangent meets the cubic in the points where $dx_2^3 = 0$, that is, all three intersections coincide in the cusp. The same must be true of a conic or other curve having the same tangent in the cuspidal point. To prove this rigorously, eliminate x_3 from



Three tangents in ordinary points.

Two tangents absorbed by a double point.

Three tangents absorbed by a cusp.

Fig. 16

the equations of cubic and conic, finding 3 roots $x_1 : x_2$ different from 0. More easily, assume that $x_1/y_1 \neq 0$ for some common point; multiply by x_1/y_1 the equation of the conic, subtract from that of the cubic, and find for the ratio x_1/x_2 a cubic equation; whence there are only 3 intersections not in the cusp $x_1 = x_2 = 0$. Therefore there are 3 tangents to the cubic from a point (y), or *the cubic with a cusp is of class* 3.

The effect of a double point or a cusp on the class of any algebraic curve of higher order is the same as on a cubic. The occurrence of a double point diminishes by 2, that of a cusp by 3, the class of any algebraic curve. The appended diagrams may serve to explain to the eye this fact, showing schematically the development of such singular points.

§ 4. Degenerate conic polars and the Hessian.

In a doubly infinite linear system of conics there occurs a simply infinite system of degenerate conics, each consisting of a pair of lines. For the equation of such a system contains linearly two parameters, λ, μ; $\Phi(x) = \varphi_1(x) + \lambda \cdot \varphi_2(x) + \mu \cdot \varphi_3(x) = 0$. Then the discriminant of this quadric $\Phi(x)$ equated to zero gives the necessary and sufficient condition for the conic to be degenerate. That condition is satisfied by an infinite number of pairs of values of λ and μ. Now conic polars of a given cubic constitute such a system, the parameters being the coordinates of the pole. What is the locus of poles of degenerate conic polars? To find the equation of that locus, equate to zero the discriminant of the quadric: $(ay)\,(ax)^2 = 0$. It will be an equation of order 3 in coordinates (y) of the pole:

$$(1) \qquad \begin{vmatrix} a_1^2(ay) & b_2b_1(by) & c_3c_1(cy) \\ a_1a_2(ay) & b_2^2(by) & c_3c_2(cy) \\ a_1a_3(ay) & b_2b_3(by) & c_3^2(cy) \end{vmatrix} = H(y) = 0.$$

Here, to avoid any ambiguity in the developed determinant, we use three letters, a, b, c indifferently as equivalent,

$$(2) \qquad f(x) \equiv (ax)^3 \equiv (bx)^3 \equiv (cx)^3.$$

This determinant (1) is known as the *Hessian* of $f(x)$; and the corresponding curve is called the Hessian curve, or simply the Hessian, of the cubic: $f(x) = 0$.

Definition. *The Hessian curve of a cubic is the locus of poles whose conic polars with respect to the cubic are degenerate.*

Degenerate polars have double points; what is their locus? With respect to a degenerate conic the polar of every point (z) of the plane passes through the double point. Let (y) denote the pole of a degenerate conic polar:

$$(3) \qquad (ay)\,(ax)^2 = 0.$$

The polar of (z) with respect to this conic is given by the equation:

$$(4) \qquad (ay)\,(az)\,(ax) = 0.$$

If (x) is taken at the double point, this equation is satisfied for every point (z) of the plane. This gives three conditions:

$$(5) \quad (ay)a_1(ax) = 0, \quad (ay)a_2(ax) = 0, \quad (ay)a_3(ax) = 0.$$

Since these three equations are linear in the coordinates (y), we can eliminate these coordinates and obtain as a determinant the function which vanishes in all double points (x) of degenerate conic polars.

$$(6) \quad \begin{vmatrix} a_1^2(ax) & a_1a_2(ax) & a_1a_3(ax) \\ b_2b_1(bx) & b_2^2(bx) & b_2b_3(bx) \\ c_3c_1(cx) & c_3c_2(cx) & c_3^2(cx) \end{vmatrix}$$

The identity of this with equation (1) is evident.

THEOREM. *The Hessian of a cubic is at the same time the locus of poles of degenerate conic polars, and also the locus of double points of those polars. The Hessian is itself a curve of third order, whose equation is given above in* (1) *and* (6).

With respect to curves of higher order than the third, these two loci are distinct, the double points of degenerate conic polars filling a curve called the Steinerian, while their poles lie on another called the Hessian.*

§ 5. Only points of the cubic are on their own polars.

The equations of the conic and linear polars of a point (y) are respectively: $(ay)(ax)^2 = 0$ and $(ay)^2(ax) = 0$. If the point (y) lies on either polar, it satisfies the equation obtained by substituting (y) for (x): $(ay)(ay)^2 = 0$ or $(ay)^3 = 0$; that is, it lies on the cubic. Conversely also every point of the cubic is on its own polars. The polars of a point (y) of the cubic touch the cubic in their pole. For the linear polar: $(ay)^2(ax) = 0$, is by its form the tangent line in (y) (III, § 3). And the tangent in (y) to the conic: $(ay)(ax)^2 = 0$, is the polar of (y) with respect to that conic; hence its equation is $(ay)[(ay)(ax)] = (ay)^2(ax) = 0$. Thus the linear

* See SALMON, Higher Plane Curves, p. 57.

polar of (y) is tangent in (y) both to the cubic and to the conic polar; all three are in contact at that point. From this we might deduce the theorem (III, § 3) that only four tangents to a cubic can be drawn from one of its points; for if (y) is on the curve, its conic polar, on which lie all points of contact of tangents from (y), touches the cubic in (y) and so can meet it at most in only four other points.

§ 6. Points of Inflexion lie on the Hessian.

At a point of inflexion the tangent contains three coincident points of intersection with the curve. If (x) be such a point, (y) any point on the tangent: $(ax)^2(ay) = 0$, then any other point of the tangent is $(x + \lambda y)$, i. e. its coordinates are $x_i + \lambda y_i$, $(i = 1, 2, 3)$. Hence the third intersection of this tangent with the curve is found by making λ satisfy the equation:

$$(1) \qquad f(x+\lambda y) = 0 = (ax)^3 + 3\lambda(ax)^2(ay) + 3\lambda^2(ax)(ay)^2 + \lambda^3(ay)^3;$$

or after dropping two zero terms and the factor λ^2,

$$3(ax)(ay)^2 + \lambda(ay)^3 = 0.$$

The point (x) is a point of inflexion if the third root of equation (1) is also equal to zero, for every arbitrary point (y) of the tangent. That is, if (x) is an inflexional point, and (y) any point of the locus:

$$(ax)^2(ay) = 0,$$

the equation: $(ax)(ay)^2 = 0$ must be satisfied. In other words, a conic is to contain all the points of a right line. This can happen only when the conic is degenerate, consisting of the tangent and some other line. But the degenerate conic: $(ax)(ay)^2 = 0$ is the conic polar of (x), therefore its pole (x) is upon the Hessian (III, § 4).

Conversely, *every point on both curves, the cubic and its Hessian, is a point of inflexion of the cubic unless it be a*

double point or a cusp. Given: $(ax)^3 = 0$ and $H(x) = 0$, to prove (x) a point of inflexion or a double point. Let (y) lie on the tangent: $(ax)^2(ay) = 0$. Since now (x) is on the Hessian, its conic polar is degenerate; but since that conic polar touches the cubic (III, § 5), two cases are possible:

(α) the contact may be improper, (x) being a double point of the cubic and an ordinary point on either line of the conic, or (x) being an ordinary (non-singular) point of the cubic, but a double point on the conic;

(β) the contact may be proper, one line of the conic being the tangent: $(ax)^2(ay) = 0$.

Case (α). If (x) is double on the cubic, the theorem needs no demonstration. If it is double on the conic, and lies on the cubic, we have the pole of a degenerate conic coinciding with the double point of its own polar. Is this possible? Calling (y) the pole, (x) the double point (III, § 4) we had the conditions of degeneracy. $(ay)(ax)a_1 = (ay)(ax)a_2 = (ay)(ax)a_3 = 0$. Here we are to have further (y) coinciding with (x), giving the conditions:

$$a_1(ax)^2 = a_2(ax)^2 = a_3(ax)^2 = 0.$$

These are however the condition (III, § 3) defining a multiple point of the cubic. Hence if (x) is a node on its own conic polar, it is a double point on the cubic.

Case (β). If the tangent in (x): $(ax)^2(ay) = 0$ forms part of the conic polar; $(ax)(ay)^2 = 0$, take (y) on the tangent. Then if $(x + \lambda y)$ denote the third intersection of tangent and cubic:

$$(a,x + \lambda y)^3 = (ax)^3 + 3\lambda(ax)^2(ay) + 3\lambda^2(ax)(ay)^2 + \lambda^3(ay)^3 = 0.$$

Three roots λ are zero, since the point (y) must lie by inclusion also on the conic polar: $(ax)(ay)^2 = 0$. Hence the tangent has three consecutive points in common with the cubic, and is so an inflexional tangent. q. e. d.

From this theorem it follows that the number of points of inflexion cannot be more than 9, since cubic and Hessian cannot meet in more than 9 distinct points. It remains to be seen whether any of these intersections coincide by reason

of proper contact, or through the occurrence of multiple points of the Hessian; and in case the cubic has a double point or a cusp, how many intersections of the Hessian are absorbed at that point.

§ 7. Corresponding or Conjugate Points on the Hessian.

To every point (x) on the Hessian corresponds another point (x') of the same curve, namely the double point or node of its degenerate conic polar (III, § 4). The equations connecting pole and node were these:

$a_1(ax)(ax') = a_2(ax)(ax') = a_3(ax)(ax') = 0$; or in extenso:

$$
\begin{aligned}
x_1 a_1^2(ax') + x_2 a_1 a_2(ax') + x_3 a_1 a_3(ax') &= 0,\\
x_1 a_1 a_2(ax') + x_2 a_2^2(ax') + x_3 a_2 a_3(ax') &= 0,\\
x_1 a_1 a_3(ax') + x_2 a_2 a_3(ax') + x_3 a_3^2(ax') &= 0.
\end{aligned}
\tag{1}
$$

From the symmetry of the equations in the two sets of coordinates (x) and (x'), each point is necessarily in reciprocal relation with the other; hence each point is the pole of a degenerate conic whose node is the other point. According to the foregoing paragraph, (case α), the following theorem is demonstrated:

THEOREM. *Two corresponding points of the Hessian can coincide only at a double point or at a cusp of the cubic.*

Can a point (x) of the Hessian have an indeterminate conjugate point? As (x) is on the Hessian, the three equations of the system (1) above are consistent, and from any two of them the ratios $x_1' : x_2' : x_3'$ are determined as ratios of 2-rowed determinants. These ratios are indeterminate when and only when all two-rowed minors of the matrix of equations (1) vanish. Of these nine conditions, three are algebraically independent. That they are not identically consistent can be shown by any one example. Take the cubic

$$f(x) = (ax)^3 = x_1^3 + x_2^3 + x_3^3 + 12x_1x_2x_3 = 0.$$

The Hessian matrix is then:

$$H(x) = -\begin{vmatrix} x_1 & 2x_3 & 2x_2 \\ 2x_3 & x_2 & 2x_1 \\ 2x_2 & 2x_1 & x_3 \end{vmatrix}.$$

Three independent minor equations are these:

$$\begin{vmatrix} x_1 & 2x_3 \\ 2x_3 & x_2 \end{vmatrix} = 0, \qquad \begin{vmatrix} x_1 & 2x_2 \\ 2x_3 & 2x_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} x_1 & 2x_2 \\ 2x_2 & x_3 \end{vmatrix} = 0.$$

No common solution can be found; for the first and third give:

$$x_1^2 x_2 x_3 = 16 x_2^2 x_3^2,$$

while the second is:

$$x_1^2 = 2 x_2 x_3.$$

Two solutions only are found, viz. $x_1 = x_2 = 0$, and $x_1 = x_3 = 0$. Neither satisfies all three of the above conditions. Since then the conditions are not always consistent, they can be consistent only when particular conditions are fulfilled. From 3 equations the coordinates (x) could be eliminated, leaving one condition only among the coefficients of the cubic. We conclude as follows.

THEOREMS. *The node of a degenerate conic polar cannot become indeterminate, that is, no two lines constituting a degenerate conic polar can coincide, unless the coefficients of the cubic satisfy a particular algebraic relation.* What this condition is will be determined most easily when the equation of the cubic has been reduced to a normal form. We notice here two things; first, that such a condition will be invariant, since projective transformation will transform the singular polar conic into a conic polar having the same singularity; second, that in case a doubly counting line does occur among the conic polars, all points of that line will lie in the Hessian, since all its points alike are double points of the conic.

§ 8. The Envelope of degenerate conic polars, or the Cayleyan.

If a point (y) moves along the Hessian, its conic polar will envelope a curve called the Cayleyan. Evidently the tangents of this Cayleyan will have a correspondence in pairs, resembling that of points on the Hessian; two tangents namely which constitute together a conic polar, and which therefore meet on the Hessian, are to be called corresponding or conjugate tangents. A reciprocal relation of the two curves will appear soon. To find the explicit equation of the Cayleyan in line coordinates, denote by (u_1, u_2, u_3) and (v_1, v_2, v_3) the coordinates of the two lines which constitute the degenerate conic polar of (y). The identical equation must subsist: $(ay)(ax)^2 \equiv (ux)(vx)$, involving the 6 equations, all linear in coordinates (y) and (v):

$$\begin{gathered} a_1^2(ay) = u_1v_1, \quad a_2^2(ay) = u_2v_2, \quad a_3^2(ay) = u_3v_3, \\ (2) \quad 2a_2a_3(ay) = u_3v_2 + u_2v_3, \quad 2a_3a_1(ay) = u_1v_3 + u_3v_1, \\ 2a_1a_2(ay) = u_2v_1 + u_1v_2. \end{gathered}$$

From these, eliminate the 6 quantities $y_1, y_2, y_3, v_1, v_2, v_3$. To avoid any ambiguity, use different letters a, b, c in the 3 columns containing coefficients of the cubic. The resulting equation for (u) is this:*

$$(3) \qquad \begin{vmatrix} a_{111} & b_{211} & c_{311} & u_1 & 0 & 0 \\ a_{122} & b_{222} & c_{322} & 0 & u_2 & 0 \\ a_{133} & b_{233} & c_{333} & 0 & 0 & u_3 \\ 2a_{123} & 2b_{223} & 2c_{323} & 0 & u_3 & u_2 \\ 2a_{113} & 2b_{213} & 2c_{313} & u_3 & 0 & u_1 \\ 2a_{112} & 2b_{212} & 2c_{312} & u_2 & u_1 & 0 \end{vmatrix} = 0.$$

For the line (v) the same equation holds, on account of the symmetry of the 6 conditions in the (u) and (v). This is then the equation of the Cayleyan.

THEOREM. *The Cayleyan is a curve of class* 3, *and its equation, like that of the Hessian, is of degree* 3 *in the coefficients of*

* Here the notation is varied slightly in the interest of brevity: a_{122} for $a_1a_2^2$, etc.

the cubic. This theorem may be stated also as follows, a useful corollary being added. *Through any arbitrary point of the plane there pass* 3 *tangents to the Cayleyan curve, that is,* 3 *lines forming parts of degenerate conic polars. Through each point of the Hessian pass therefore two lines constituting one conic polar, and only one line forming part of a different degenerate conic polar.*

As related to the cubic, the Hessian and the Cayleyan were defined by means of the system of conic polars. Every projective transformation of the cubic changes these conic polars into those of the transformed cubic, and the degenerate conics of the one system into those of the transformed system; therefore the Hessian and Cayleyan must pass by the same projection into the Hessian and Cayleyan of the transformed cubic. Briefly expressed: *the Cayleyan and Hessian are invariants of collineation or projection.*

§ 9. The four poles of a line.

THEOREM. *If a point* (x) *lies on the conic polar of* (y), *then* (y) *lies on the linear polar of* (x), *and conversely.*

PROOF. The equation: $(ay)(ax)^2 = 0$ which embodies the hypothesis, asserts also the conclusion: $(ax)^2(ay) = 0$, where $(az)^3 = 0$ is the equation of the cubic.

THEOREM. *The conic polars of all points on a straight line form a pencil or sheaf of conics, having four points in common.* By virtue of the foregoing theorem this is equivalent to the following: *every given line is the linear polar of four points with respect to a cubic curve. Briefly: every line has four poles.*

PROOF. If (x) and (y) are two points of a line, every other point of that line has coordinates of the form: $t_i = x_i + \lambda y_i$ $(i = 1, 2, 3)$. The conic polar of (t) is given by the equation:

$$(at)(az)^2 = (ax)(az)^2 + \lambda(ay)(az)^2 = 0,$$

and this is the equation of any conic containing the four points common to the loci:

$$(ax)(az)^2 = 0, \qquad (ay)(az)^2 = 0;$$

that is to say, the four points common to the conic polars of (x) and (y). Varying λ in the above equation, we have corresponding to all points of the line $\overline{xy}$, all conics of a pencil or sheaf. Further, each of the four common points, as it lies on the conic polar of every point of the line $\overline{xy}$, must have a linear polar containing all points of that line. In other words, all four points have the line $\overline{xy}$ for linear polar, and the line has the four poles. q. e. d.

If one of its poles lies on the line itself, the line is tangent to the cubic (III, § 5). If two poles lie on the line, since no line can touch the cubic in two points, it must be doubly tangent in one point; that is, only inflexional tangents of a non-singular cubic contain two of their own poles.

§ 10. The Poloconic of a line.

THEOREM. *If a point describes a line, its linear polar envelopes a conic, which is called the poloconic of the line, (by Salmon, the "Polar conic of the line"). This conic is also the locus of poles of conic polars that touch the line.* To obtain the equation, use for convenience an undetermined parameter as in the ordinary method for envelopes. The point (y) is to describe the line: $(uy) = 0$, and we seek the envelope of its linear polar: $(ay)^2(ax) = 0$. Call the parameter ϱ; the conditions are:*

$$\begin{aligned} a_1(ay)(ax) &= (ax)\{a_{11}y_1 + a_{12}y_2 + a_{13}y_3\} = \varrho u_1, \\ b_2(by)(bx) &= (bx)\{b_{21}y_1 + b_{22}y_2 + b_{23}y_3\} = \varrho u_2, \\ c_3(cy)(cx) &= (cx)\{c_{31}y_1 + c_{32}y_2 + c_{33}y_3\} = \varrho u_3, \\ (uy) &= u_1y_1 + u_2y_2 + u_3y_3 = 0. \end{aligned}$$

Elimination of y_1, y_2, y_3, ϱ gives for the point of contact (x) the equation:

$$\begin{vmatrix} a_{11}(ax) & a_{12}(ax) & a_{13}(ax) & u_1 \\ b_{21}(bx) & b_{22}(bx) & b_{23}(bx) & u_2 \\ c_{31}(cx) & c_{32}(cx) & c_{33}(cx) & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} = 0.$$

* The notation here will be understood by comparison with equation (3).

The same equation would have resulted if we had required the line (u) to touch the conic: $(ax)(ay)^2 = 0$, for it is the discriminant of the conic bordered with the coordinates of the line. The equation constitutes therefore the proof of the above theorem.

Poloconics form a doubly infinite system, since the lines to which they correspond are a doubly infinite system. Unlike the conic polars of points, the poloconics do not form a linear system; for the coordinates (u) of the variable line enter the equation in the second degree. The poloconics have remarkable properties, particularly in relation to the Hessian.

§ 11. Intersections of a line and its Poloconic.

Let a line u intersect its poloconic C in two points A, B. Call the linear polar and conic polar of A respectively a and ka. We shall find that a is tangent to C in B. If a does not meet u in B, let P denote its intersection. Now a is the polar line of A also with respect to ka, and ka must touch the line u because its pole A is on the poloconic C. Therefore the point of contact of u with ka must be P, where the polar of A cuts the tangent u from A to ka. Since therefore P is on the conic polar of A, A is on the linear polar of P, and for the converse reason A is on the conic polar of P. But the point where the linear polar of P meets its conic polar is a point of contact of a tangent from P, hence PA or u is tangent to the conic polar of P. Therefore by (III, § 10), the point P is on the poloconic of u, or C. P must be then either A or B. If it were A, it must lie on its own polars and so on the cubic. In general therefore P coincides with B; and a the tangent through P is tangent in B. Thus is proven the following theorem.

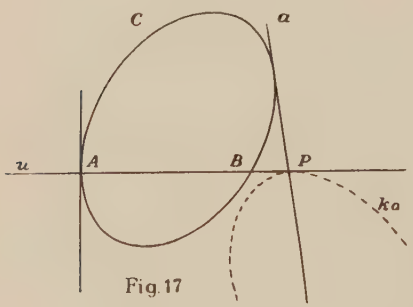


Fig. 17
Represents false hypothesis.

THEOREM. *If A and B denote the intersections of a line with its poloconic, the conic polar of A touches the line in B, and reciprocally: also the linear polar of A touches the poloconic in B and reciprocally.*

§ 12. Poloconics that touch their corresponding lines.

The poloconic C of a line u meets its line in two points, A and B. *If A lies on the cubic, B must coincide with it and u must be tangent to the cubic.* For the line u is tangent to the conic polar of A, (III, § 10), but the conic polar of A is tangent to the cubic because A is a point of the cubic (III, § 5), and contains A. Thus the line u passing through A and touching a conic which at the point A is tangent to the cubic, must itself touch the cubic at A, and is therefore identical with the linear polar of A. But this linear polar is tangent to C, hence it is the tangent in A, and therefore B and A coincide; as stated above.

Conversely if B coincides with A, so that the line u is tangent to its poloconic, it must touch the cubic also in A. For A will then lie on its own polars, as they both pass through B (III, § 11), and so must lie upon the cubic (III, § 5); whence by the above direct proof, u is tangent to the cubic. The following theorem is therefore proven.

THEOREM. *All tangents of the cubic are tangent to their respective poloconics, and conversely every line which touches its poloconic touches also the cubic in the same point.* It follows that the cubic is a part of the envelope of poloconics which are tangent to their own lines. The theorem gives a means of expressing readily the equation of the cubic in line coordinates.

§ 13. The tangential equation of the cubic.

As an auxiliary, the equation of a conic in line coordinates may first be reduced to a condensed form by the abbreviated notation already introduced (III, § 1). The condition that two lines (a) and (c) shall be conjugate with respect to a conic:

$(Ax)^2 = 0 = (Bx)^2 = (Cx)^2$, is obtained by equating to zero the discriminant bordered with coordinates u, v; it is:

$$(1) \qquad F(u,v) \equiv \begin{vmatrix} A_1^2 & A_1A_2 & A_1A_3 & v_1 \\ B_1B_2 & B_2^2 & B_2B_3 & v_2 \\ C_1C_3 & C_2C_3 & C_3^2 & v_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} = 0.$$

Expanding this we may write it: $F(u,v) \equiv \sum u_1v_1 \cdot B_2C_3(B_2C_3) - \sum u_2v_3 \cdot A_1B_2 \cdot (A_1B_3)$.

Replacing B, C in the first summation by A, B, and arranging factors:

$$(2) \qquad F(u,v) \equiv \sum u_1(A_2B_3) \cdot v_1A_2B_3 + \sum u_2(A_3B_1) \cdot v_3A_1B_2.$$

Exchanging letters A, B will not alter the meaning, but will evidently change the sign of the determinants like $(A_2B_3) = A_2B_3 - A_3B_2 = -(B_2A_3)$. One of these occurs in every term, hence we find:

$$(3) \qquad F(u,v) \equiv -\sum u_1(A_2B_3) \cdot v_1A_3B_2 - \sum u_2(A_3B_1) \cdot v_3A_2B_1.$$

Add equations (2) and (3), and combine corresponding terms:

$$\begin{aligned} 2F(u,v) &\equiv \sum u_1(A_2B_3) \cdot v_1(A_2B_3) + \sum u_2(A_3B_1) \cdot v_3(A_1B_2) \\ &\equiv \left[\sum u_1(A_2B_3)\right] \cdot \left[\sum v_1(A_2B_3)\right] \\ &\equiv (u_1A_2B_3)(v_1A_2B_3) \equiv (ABu)(ABv). \end{aligned}$$

THEOREM. *The condition that two lines* (u), (v) *shall be conjugate with respect to a conic whose equation is denoted by* $(Ax)^2 - (Bx)^2 = 0$ *may be written briefly in the form:* $(ABu)(ABv) = 0$.

The condition that a line (u) *shall be conjugate to itself, that is, that it shall be tangent to the conic, or the tangential equation of the conic is in abbreviated notation the following:*

$$(ABu)(ABu) \text{ or } (ABu)^2 = 0.$$

The equation of the poloconic of a line (u) with respect to the cubic: $(ax)^3 = 0$ was given by a bordered discriminant

(III, § 10); it is namely the tangential equation of the conic: $(ax)(ay)^2 = 0$, interpreted however as having constant (u) and variable (x). Hence its equation in condensed notation is this:

$$(ax)(bx)(abu)^2 = 0.$$

If the line (u) touches this its poloconic, it is a tangent of the cubic. The condition is again a bordered discriminant. We use the further abbreviation $(\theta x)^2(\vartheta u)^2$ for the above form, quadric in the two sets of coordinates (u) and (x); and since coefficients will often occur to the second degree, we use two distinguishable letters θ and θ', thus:

$$(ax)(bx)(abu)^2 \equiv (\Theta x)^2(\vartheta u)^2 \equiv (\Theta' x)^2(\vartheta' u)^2 = 0.$$

The discriminant of this quadric in (x), bordered with coordinates (u), gives the desired equation:

$$(1) \qquad (\vartheta u)^2(\vartheta' u)^2(\Theta\Theta' u)^2 = 0.$$

To find the explicit formula in letters a and b, c and d, coefficients of the fundamental cubic, notice that the explicit form $(ax)(bx)(abu)^2$ is derivable from the abbreviated form by the use of differential operators:

$$(2) \quad (ax)(bx)(abu)^2 = \frac{1}{4}\left(a\frac{\partial}{\partial\theta}\right)\left(b\frac{\partial}{\partial\theta}\right)\left((ab)\frac{\partial}{\partial\vartheta}\right)^2[(\theta x)^2(\vartheta u)^2].$$

(Here the expression $\left((ab)\frac{\partial}{\partial\vartheta}\right)$ signifies $\sum(a_1b_2)\frac{\partial}{\partial\vartheta_3}$). Applying this operator twice to the equation (1), i. e. replacing the sign θ by a and b, the sign θ' by c and d, in the complete operator

$$\frac{1}{16}\left(a\frac{\partial}{\partial\theta}\right)\left(b\frac{\partial}{\partial\theta}\right)\left((ab)\frac{\partial}{\partial\vartheta}\right)^2\cdot\left(c\frac{\partial}{\partial\theta'}\right)\left(d\frac{\partial}{\partial\theta'}\right)\left((cd)\frac{\partial}{\partial\vartheta'}\right)^2,$$

we find the explicit equation:

$$(3) \qquad (abu)^2(cdu)^2\{(acu)(bdu)+(adu)(bcu)\} = 0.$$

This equation, (1) or (3), is the tangential equation of the non-singular cubic. For it contains the line coordinates to the sixth degree; if combined with a linear equation: $u_1\alpha_1 + u_2\alpha_2 + u_3\alpha_3 = 0$ which requires the line (u) to pass through the point (α), it must give six solutions. But the curve of third order is in general of class six, as already proven (III, § 3) by the aid of first polar. Since the cubic is part of this envelope, and is itself of class 6, it must be the complete envelope just found.

§ 14. Poloconics touch the Hessian.

By the following loose argument it can be made to appear probable that a poloconic is tangent to the Hessian. Let the line u cut the Hessian in points A_1, A_2, A_3, while C, the poloconic of u, cuts it in six points $C_1, C_2, \cdots, C_6$. Each of these six points is on both Hessian and poloconic, and so has a conic polar which is degenerate and tangent to the line u. Of such there are only three, those whose nodes are at A_1, A_2, A_3, (in proper contact). It seems therefore probable that the six points $C_1, C_2, \cdots, C_6$, may coincide in pairs; and especially so since we know three points in which Hessian and poloconic must intersect, namely the three poles of degenerate conics whose nodes are A_1, A_2, A_3, i. e. the three points of the Hessian corresponding to A_1, A_2, and A_3 (III, § 7). For these latter are poles of conics having nodes, and so improper contacts, on the line u. Call the point corresponding to A_1, B_1. The conic polar of A_1 has a node in B_1, therefore the linear polar of A_1 passes through B_1. This linear polar therefore meets the poloconic in B_1, and touches the poloconic at some point (III, § 10), hence they are tangent at B_1. If now this same linear polar is tangent to the Hessian in B_1, so is also the poloconic. This must be formally investigated.

THEOREM. *The linear polar of a point A on the Hessian is tangent to the Hessian in the point B corresponding to A.*

PROOF. If the coordinates of A are $\alpha_1, \alpha_2, \alpha_3$; of B, $\beta_1, \beta_2, \beta_3$, they are connected through the equation of the cubic: $(ax)^3 \equiv (bx)^3 \equiv (cx)^3 = 0$, by the relations (III, § 7):

$$a_1(a\alpha)(a\beta) = 0, \quad b_2(b\alpha)(b\beta) = 0, \quad c_3(c\alpha)(c\beta) = 0.$$

These 3 relations are consistent, since B is on the Hessian; and any 2 of them will give the coordinates (α) as two-rowed minors of the Hessian:

$$H(\beta) = \begin{vmatrix} a_{11}(a\beta) & a_{12}(a\beta) & a_{13}(a\beta) \\ b_{12}(b\beta) & b_{22}(b\beta) & b_{23}(b\beta) \\ c_{13}(c\beta) & c_{23}(c\beta) & c_{33}(c\beta) \end{vmatrix} = 0.$$

Call the minors complementary to the first row: H_{11}, H_{12}, H_{13}; and similarly for the other rows. From the identities of this symmetric determinant we have: $H_{12} = H_{21}$, $H_{13} = H_{31}$, $H_{23} = H_{32}$. Since then the coordinates α_1, α_2, α_3 are proportional to the minors of any two rows or columns, we have in the following two arrays corresponding terms proportional.

$$\begin{Bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \alpha_1^2 & \alpha_1\alpha_2 & \alpha_1\alpha_3 \\ \alpha_2\alpha_1 & \alpha_2^2 & \alpha_2\alpha_3 \\ \alpha_3\alpha_1 & \alpha_3\alpha_2 & \alpha_3^2 \end{Bmatrix}$$

To find the equation of the tangent in B to the Hessian, apply the polar operator $\left(z \frac{\partial}{\partial \beta}\right)$ to $H(\beta)$. This is most readily done by taking all constituents of any one row together; (since they are contained *linearly* in the expanded determinant, this is permissible). The equation of this tangent is therefore:

$$\begin{Bmatrix} a_{11}(az)\cdot H_{11} + a_{12}(az)\cdot H_{12} + a_{13}(az)\cdot H_{13} \\ + b_{21}(bz)\cdot H_{21} + b_{22}(bz)\cdot H_{22} + b_{13}(bz)\cdot H_{23} \\ + c_{31}(cz)\cdot H_{31} + c_{32}(cz)\cdot H_{32} + c_{33}(cz)\cdot H_{33} \end{Bmatrix} = 0.$$

Replacing the quantities H_{ik} by the proportionate products $\alpha_i\alpha_k$, there results:

$$\sum a_{ik}\alpha_i\alpha_k(az) = 0, \text{ or } (a\alpha)^2(az) = 0,$$

which is exactly the linear polar of the point A, q. e. d. This completes the proof of the theorem expected.

THEOREM. *The poloconic of an arbitrary line is tangent in* 3 *points to the Hessian of the fundamental cubic, and these points of contact are the double points of the conic polars of those* 3 *points in which the arbitrary line meets the Hessian. Of the three points of contact, two may be chosen arbitrarily, and the third is thereby determined.* For if B_1 and B_2 are given, their corresponding points on the Hessian, A_1 and A_2, fix the line u and its third intersection A_3, which latter has for corresponding point the third contact B_3.

The poloconics of a cubic are now seen to be a system of conics triply tangent to its Hessian. If the same curve can be Hessian to several cubics, it would have corresponding to each of those cubics a system of triply tangent conics. By (II, § 6) and (III, § 3) the number of different systems cannot be more than three. The question arises then whether the Hessian of a given cubic may possibly have the same relation to two others, and whether indeed every cubic may not be the Hessian of three different cubics. (See V, § 3.)

If two points of contact B_1 and B_2 coincide in $\bar{B}$, B_3 is the point corresponding to the tangential of the coincident points A_1 and A_2, say of $\bar{A}$; and the conic hyperosculates the Hessian in $\bar{B}$, having ordinary contact in B_3. In the system of poloconics there is at every point B one hyperosculating conic. If further the points B and B_3 should coincide in $\bar{B}$, the points $\bar{A}$, A_3 must also coincide, making the line u tangent to the Hessian at a point of inflexion; therefore the tangent in B must pass through a point of inflexion. Points B on a cubic, in which a conic can have its six intersections coincident, are called *sextactic points. On a Hessian, the corresponding points of the nine inflexions are sextactic points.*

CHAPTER IV.

Inflexions and harmonic polars.

§ 1. The harmonic polar of an inflexion.

A cubic is cut by the conic polar of a point wherever a tangent can be drawn from the point to the cubic. If the point Y is on the cubic: $(ay)^3 = 0$, from it only 4 tangents can be drawn (III, § 3). It is possible to give a familiar geometric definition of the polar which determines these 4 points of contact. If any secant YZ be laid through Y, it meets the cubic in two additional points X_1 and X_2 given by the equation:

$$(ax)^3 = (a, y + \lambda z)^3 = (ay)^3 + 3\lambda(ay)^2(az) + 3\lambda^2(ay)(az)^2 + \lambda^3(az)^3 = 0,$$

or

$$0 = (ay)^2(az) + \lambda(ay)(az)^2 + \frac{\lambda^2}{3}(az)^3.$$

Here $\lambda : 1$ is the ratio of the distances $YX : XZ$. If the roots are λ_1, λ_2, and we assume Z such as to make the coefficient of λ zero: $(ay)(az)^2 = 0$, then $\lambda_1 + \lambda_2 = 0$, i. e.

$$\frac{YX_1}{X_1Z} + \frac{YX_2}{X_2Z} = 0, \quad \therefore \quad \frac{YX_1 \cdot X_2Z}{YX_2 \cdot X_1Z} = -1.$$

This asserts that the points X_1 and X_2 are harmonic to the points Y, Z; that is, that Z is the harmonic conjugate of Y with respect to the points where a secant through Y meets the curve.

THEOREM. *If on each secant of the cubic through one of its points Y we determine the harmonic conjugate of that point Y with respect to the other two intersections, the locus of these harmonic points is the conic polar of Y.* When X_1 coincides with X_2 at a contact, the harmonic point Z unites with them; and if X_1 approaches the pole Y along the curve, Z must come

to coincide with Y at the same instant, so that the conic polar of Y is tangent in Y to the cubic (III, § 5).

DEFINITION. *Let Y denote one of the 9 points of inflexion, then its conic polar separates,* (III, § 6), *into two lines, one of which is the inflexional tangent at Y. The second component is called the harmonic polar of the point of inflexion.* If two points of the harmonic polar are known, the line of course is determined. As it is a part of the conic polar, it divides harmonically all secants through its corresponding inflexion. If then YAB and YCD are two inflexional lines (II, § 7), the two points harmonically separated from Y by A and B, and by C and D, respectively, will determine the harmonic polar of the inflexional point Y. The nine points of inflexion and their nine harmonic polars are systems exhibiting a peculiar arrangement, now to be examined.

§ 2. Lines connecting the nine inflexions.

Disregarding at first the distinction of real and imaginary, consider the lines joining the 9 inflexions of a cubic in sets of 3. If we wish to assign to the points of inflexion the numbers 1, 2, 3, . . . , 9, begin by assigning to any two among them the marks 1 and 2. The line (1, 2) contains a third inflexion, which may be denoted by 3. Connect each of the first three with another, e. g. with one denoted by 7; the lines (1, 7), (2, 7), and (3, 7) contain three additional inflexions, which shall be called respectively 4, 6 and 5. Now the line (123), the inflexional tangent in 7, and the line (4, 5) constitute a cubic through eight points; another through the same eight points consists of the lines (174), (276), (375); hence the ninth point must also be common to both, i. e. the line (4, 5) must contain the point 6. (See II, § 5). The line joining 7 to one of the remaining points as 8, cannot contain 1, 2, 3, 4, 5, or 6, hence it must contain the sole remaining point 9. The line (123), (456), (789) are seen to contain all the inflexions; the second and third were reached in a definite manner from the first, we conclude therefore that any one line joining three inflexions individuates two others contain-

ing the remaining six inflexions. A given inflexion, as 1, lies on four such lines, hence there are eight others, or twelve in all.

To make a complete scheme of these lines, notice that when three are found: (123), (456), (789), a line joining a point of the first to a point of the second cannot contain a second point of either, but must therefore have its third intersection at an inflexion on the third line. As the line (1, 4) contains 7, let (1, 5) contain 9, the (1, 6) must contain 8. These lines may be indicated by the following scheme, where for convenience the points 7 and 1 appear in two positions: Since already three lines pass through 5, a fourth must contain 2 and 8. Inserting that, we observe three lines through 2, and determine a fourth to be (249). The same reason gives us the remaining lines (483) and (369). To represent in one scheme all twelve, each by three digits in a line, we must repeat certain digits, as in the following diagram. This complete scheme shows twelve lines, easily remembered in a square array as three horizontal, three vertical, and the other six corresponding to the six terms in the expansion of a determinant.

Definition. *Each set of three lines containing all nine inflexional points of a cubic is called an inflexional triangle.* The foregoing diagram, strictly from previous propositions, verifies the following theorem.

Theorem. *The nine inflexional points of a cubic lie three and three on twelve lines constituting four inflexional triangles.*

§ 3. The intersections of harmonic polars.

Denote the harmonic polars of the inflexions 1, 2, 3, ... by h_1, h_2, h_3, ... Look at any inflexional triangle, as (123), (456), (789). Secants drawn from 1 to 4, 5, and 6 contain points on the third side, viz. 7, 9, and 8 respectively (Fig. 18) Accordingly the harmonic conjugates of the point 1 with

respect to 4, 7, to 5, 9, and to 6, 8, lie on a line passing through the intersection of (456) and (789). That is to say: *the harmonic polar of a point on any side of an inflexional triangle must pass through the vertex opposite to that side*: The harmonic polar of each inflexion must pass then through four vertices, one in each triangle; and each vertex lies on three harmonic polars, those whose inflexions lie on the opposite side of the triangle. So the harmonic polars pass by threes through the vertices of the four triangles; and as the 12 sides pass by fours through the inflexions, so 12 vertices lie by fours upon the polars.

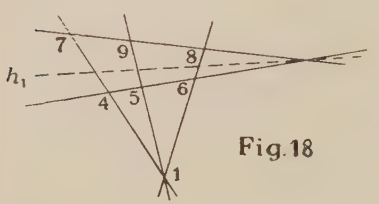


Fig. 18

The same facts are also stated thus. Each inflexional triangle lies in perspective with any other from six different centers, the vertices of the other two inflexional triangles.

§ 4. The number of real inflexions.

To find how many inflexions are real we may study the equations of cubic and Hessian, or we may look more closely at the relative positions of the inflexions on the sides of an inflexional triangle. Take first the latter method. Let the sides (123) and (456) meet in a point P (Fig. 19). From the inflexion 7 (IV, § 2) the points P, 4, 5, 6 of the one line can be projected upon P, 1, 3, 2 of the other line, as is

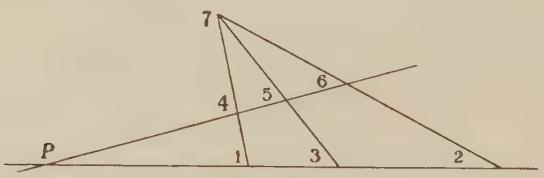


Fig. 19

seen by reference to the list of inflexional lines. Similarly from the point 8, P, 4, 5, 6 are projected upon P, 3, 2, 1; and from the point 9, upon P, 2, 1, 3. But by projection the cross-ratio of any four points is not altered (Salmon,

Conics, p. 55). Therefore we have the three equal cross-ratios $(P132) = (P213) = (P321)$. Denote one of these by x, and express the others in terms of x:

$$x = \frac{\overline{P1} \cdot \overline{32}}{\overline{P2} \cdot \overline{31}}, \qquad \frac{\overline{P2} \cdot \overline{13}}{\overline{P3} \cdot \overline{12}} = \frac{1}{1-x},$$

$$\frac{\overline{P3} \cdot \overline{21}}{\overline{P1} \cdot \overline{23}} = 1 - \frac{1}{x}.$$

These being equal, we have: $x = \frac{1}{1-x}$ and $x = 1 - \frac{1}{x}$, and from both: $x^2 - x + 1 = 0$.

The roots of this equation are imaginary,—in fact, they are the imaginary cube roots of -1. Four points whose cross-ratio is one of these imaginary cube roots of -1 are called an equianharmonic quadruple: this gives us a theorem.

THEOREM. *Upon each side of any inflexional triangle the cross-ratio of the three inflexions and either one of the vertices is an imaginary number, equal to one of the cube roots of negative unity. In other words, these four points constitute always an equianharmonic quadruple.*

We know already (I, § 5) that on a cubic without double point three of the inflexions are real, and that these lie on a line (II, § 7). That line belongs to one inflexional triangle, but the two vertices of the triangle that lie on that real line cannot be real points, for four real points would have a real cross-ratio. These two vertices being imaginary, the other sides of the triangle are imaginary. An imaginary line cannot contain more than one real point; hence, of the three inflexions on each imaginary side, at least two are imaginary points. If one of these imaginary points of inflexion be joined to each of the real inflexions, the three imaginary lines resulting must contain, each of them, two imaginary inflexional points. Accordingly all the inflexions on each of these imaginary sides are imaginary. This proves the following proposition.

THEOREM. *A real cubic which has no double-point or cusp has six imaginary and three real points of inflexion.*

The six imaginary points of inflexion must be three pairs of conjugate imaginaries, for they lie upon a real cubic and its real Hessian. Therefore the two imaginary lines just now considered must each contain one point from each pair of conjugate inflexions, because a line through two points of any pair would be necessarily a real line. How many then of the remaining nine inflexional lines are real? Evidently three, those connecting pairs of conjugate points. Do these three real lines pass through all three real inflexions, or do two or more lie through the same real point? Call the pairs of conjugates A_1, A_2; B_1, B_2; C_1, C_2. The real point of $(A_1 B_1)$ lies also on $(A_2 B_2)$, that of $(A_1 B_2)$ on $(A_2 B_1)$, etc. Through each real inflexion must pass therefore an *even* number of imaginary (conjugate) lines of inflexion; these cannot be more than 3 in number at any one real point (IV, § 2), nor can they be zero in number at any point, since then there would be 3 at each of the other two real points, hence there are exactly 2 at each point. Therefore the three real lines joining conjugate imaginary inflexions contain all three real inflexions, and constitute an inflexional triangle.

THEOREM. *Of the four inflexional triangles of a cubic, only one consists of three real lines. One has one real side and one real vertex (opposite the real side). The other two have all their sides and vertices imaginary, the sides of the one being respectively conjugate to the sides of the other.* Exchanging the words point and line, vertex and side, a corresponding dual theorem can be stated for the intersections of the nine harmonic polars, that is, for the inflexional triangles considered as sets of 3 points.

§ 5. Equations of inflexional lines.

When we assume the one real inflexional triangle for a triangle of reference, the equation of the cubic will have fewer than 10 terms, 6 must disappear. Through any one vertex will pass the harmonic polars of the inflexions on the opposite side (IV, § 3); those are components in the conic polars of the points of inflexions, therefore the linear polar of each

vertex goes through all three inflexions on the opposite side (III, § 9), and is itself that opposite side. Hence the second polar

$$\text{of } (y_1 = 1,\ y_2 = y_3 = 0) \text{ is } x_1 = 0,$$
$$\text{of } (y_1 = 0,\ y_2 = 1,\ y_3 = 0) \text{ is } x_2 = 0,$$
$$\text{of } (y_1 = y_2 = 0,\ y_3 = 1) \text{ is } x_3 = 0.$$

Therefore the only terms containing x_1^2, x_2^2, x_3^2 are those in x_1^3, x_2^3, x_3^3. For any term $x_1^2 x_2$ would give rise, in the linear polar of (1, 0, 0), to a term in x_2. After these three terms a fourth is admissible, containing $x_1 x_2 x_3$. The cubic equation must have then the form

$$ax_1^3 + bx_2^3 + cx_3^3 + 6dx_1x_2x_3 = 0.$$

To simplify further, take

$$x_1 = \frac{x_1'}{\sqrt[3]{a}},\ x_2 = \frac{x_2'}{\sqrt[3]{b}},\ x_3 = \frac{x_3'}{\sqrt[3]{c}};\quad m = \frac{-d}{\sqrt[3]{(abc)}}.$$

Then removing primes we have Hesse's normal form of the cubic equation:

I. $$F(x) = x_1^3 + x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0.$$

Each side of the triangle of reference now meets the curve in points of inflexion. The coordinates of the nine inflexions are therefore:

$$x_1 = 0,\ x_2/x_3 = -1,\ -\omega,\ \text{or } -\omega^2; \quad (\omega^3 = +1)$$
$$x_2 = 0,\ x_3/x_1 = -1,\ -\omega,\ \text{or } -\omega^2;$$
$$x_3 = 0,\ x_1/x_2 = -1,\ -\omega,\ \text{or } -\omega^2.$$

Noticing what sets of three lie on right lines, we have an arrangement corresponding to that in (IV, § 2).

$$(0,\ 1,\ -\omega) \quad (0,\ 1,\ -\omega^2) \quad (0,\ 1,\ -1)$$
$$(-\omega,\ 0,\ 1) \quad (-\omega^2,\ 0,\ 1) \quad (-1,\ 0,\ 1)$$
$$(1,\ -\omega,\ 0) \quad (1,\ -\omega^2,\ 0) \quad (1,\ -1,\ 0)$$

This table giving explicitly the coordinates of all inflexions can be abbreviated to a mnemonic scheme by using only two indices for each point. The first index shall denote the side of the triangle, 1, 2, 3 according as $x_1 = 0$, $x_2 = 0$, or $x_3 = 0$. For the second index, consider the coordinates in cyclic order (1 2 3) and take the ratio of the second after a zero to the first after the zero: it is $-\omega^1$, $-\omega^2$, or $-\omega^3$ ($\omega^3 = 1$). Write the exponent of this power of ω for the second index. Thus in the second column, third row, occur the indices 3, 2, showing that the point is on the third side ($x_3 = 0$), and is, so to speak, the second inflexion on that side, i. e. $x_2/x_1 = -\omega^2$. The table becomes the following

11	12	13
21	22	23
31	32	33

Any three that lie on a line have for the sums of first indices and of second indices respectively multiples of 3. This resembles closely a scheme which will result later from the application of elliptic functions.

The equations of inflexional lines may be calculated from the coordinates of the inflexions, or they may be obtained by factoring. If the equation of the curve has the form

$$pqr = cx_1x_2x_3$$

when referred to the real inflexional triangle, we know that $p = 0$, $q = 0$, and $r = 0$ are inflexional lines, for they meet the curve in its intersections with $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, and those are exactly the nine inflexions. Now we can bring the equation of the cubic into this factorable form in three ways.

$$x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 = 3(2m-1)x_1x_2x_3,$$

$$x_1^3 + x_2^3 + x_3^3 - 3\omega x_1x_2x_3 = 3(2m-\omega)x_1x_2x_3,$$

$$x_1^3 + x_2^3 + x_3^3 - 3\omega^2 x_1x_2x_3 = 3(2m-\omega^2)x_1x_2x_3.$$

The left-hand member is factored by inspection. The equations of the sides of the other three inflexional triangles are therefore these:

$$\text{II.} \qquad x_1+x_2+x_3=0, \qquad x_1+\omega x_2+\omega^2 x_3=0,$$
$$x_1+\omega^2 x_2+\omega x_3=0;$$

$$\text{III.} \qquad x_1+x_2+\omega^2 x_3=0, \qquad x_1+\omega^2 x_2+x_3=0,$$
$$\omega^2 x_1+x_2+x_3=0;$$

$$\text{IV.} \qquad x_1+x_2+\omega x_3=0, \qquad x_1+\omega x_2+x_3=0,$$
$$\omega x_1+x_2+x_3=0.$$

Only one of these lines is real, the first in II; the others in II are conjugate imaginary lines, and so have a real intersection. The 3 lines of triangle III are seen to be conjugate respectively to those of triangle IV, as the constants ω and ω^2 are conjugate.

The lines of II intersect the lines of III in 9 distinct points. Hence Hesse's explicit equation (I) is the evidence that *there are cubic curves with equations involving* 9 *arbitrary constants, which meet their Hessians in* 9 *distinct points.*

§ 6. The nine points of inflexion are inflexions of every cubic which contains them.

Before attacking the Hessian by an algebraic method, it is of advantage to notice a geometrical fact. Secants from any inflexional point, meeting the curve in two additional points, are cut by the conic polar in a fourth point harmonic to the point of inflexion. Four such secants from one inflexional point are inflexional lines of the cubic, and the four harmonic points lie in a right line, the harmonic polar of the first inflexion (IV, § 1). Now the Hessian also contains these 9 inflexions; hence any one of the 9 must have four points of its conic polar *with respect to the Hessian* lying in a line. Therefore this conic polar is degenerate, i. e. the

point is an inflexion of the Hessian. The same argument is valid for any other cubic containing the same nine points.

THEOREM. *All cubics passing through the* 9 *intersections of a cubic and its Hessian have these intersections for inflexional points; and all these curves have the same system of* 9 *harmonic polars.*

In particular the Hessian has the same inflexional points, though not the same inflexional tangents, as its fundamental cubic, and accordingly the Hessian of the Hessian passes through the same 9 *points.*

That the cubic and its Hessian have in general different inflexional tangents is apparent from this, that if they have one such in common, they must have at least 11 points in common (the contact at an inflexion being counted as 3 points), and so must coincide throughout. Then each point A on the cubic, as being on the Hessian, must have for linear polar the tangent in some other point B of the Hessian (III, § 14), i. e. of the cubic; but the linear polar of A is the tangent in A, and cannot touch elsewhere unless the curve is degenerate. Hence no proper cubic coincides with its Hessian or has with it a common inflexional tangent.

Let us calculate the equation of the Hessian $H(x)$ from the normal form: $f(x) = x_1^3 + x_2^3 + x_3^3 - 6\,m\,x_1x_2x_3 = 0$.

$$(1) \qquad H(x) = \begin{vmatrix} x_1 & -mx_3 & -mx_2 \\ -mx_3 & x_2 & -mx_1 \\ -mx_2 & -mx_1 & x_3 \end{vmatrix}$$

$$= -m^2(x_1^3 + x_2^3 + x_3^3) + (1 - 2m^3)x_1x_2x_3.$$

If m is not zero, divide by $-m^2$, so that the equation of the Hessian curve becomes:

$$(2) \qquad x_1^3 + x_2^3 + x_3^3 - \frac{1-2m^3}{m^2}x_1x_2x_3 = 0;$$

or, setting $\mu = \dfrac{1-2m^3}{6m^2}$,

it may be written:

$$x_1^3 + x_2^3 + x_3^3 - 6\mu x_1 x_2 x_3 = 0. \tag{3}$$

If the cubic and its Hessian were to coincide, we must have

$$m = \mu = \frac{1 - 2m^3}{6m^2} \tag{4}$$

$$\begin{aligned} &\therefore\ 1 - 8m^3 = 0 \text{ or else } m = \infty, \\ &\therefore\ m = \frac{1}{2},\ \frac{\omega}{2},\ \text{or } \frac{\omega^2}{2};\ \text{unless } m = \infty. \end{aligned} \tag{5}$$

Substituting these four values, we have the four cubics:

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 - 3x_1 x_2 x_3 &= 0, \\ x_1^3 + x_2^3 + x_3^3 - 3\omega x_1 x_2 x_3 &= 0, \\ x_1^3 + x_2^3 + x_3^3 - 3\omega^2 x_1 x_2 x_3 &= 0, \\ x_1 x_2 x_3 &= 0. \end{aligned}$$

These are the four sets of three inflexional lines otherwise found in § 5. This amounts to a verification of the above statement, that no proper cubic coincides with its Hessian.

The curves: $f(x) = 0$ and $H(x) = 0$ being in general two distinct cubics through the 9 inflexions, and their intersections lying no more than 3 on any line, no more than 6 on any conic, every other cubic through the same 9 points: $F(x) = 0$, must have its equation linearly compounded of the first two (II, § 3): $F(x) \equiv \lambda_1 \cdot f(x) + \lambda_2 \cdot H(x)$.

Definition. *The simply infinite totality of all cubics having in common their nine points of inflexion is called a syzygetic sheaf of cubics. If one of them, having no double point or cusp, has the equation* $f = 0$, *and its Hessian, the equation* $H = 0$, *every curve in the syzygetic sheaf has an equation of the form:*

$$\lambda_1 \cdot f + \lambda_2 \cdot H = 0.$$

Chapter V.

Invariants of Hesse's normal form. Geometrical applications.

§ 1. An invariant relation among a cubic, its Hessian, and its pro-Hessian.

Definition. *If a cubic H is the Hessian of another cubic C, then C is termed a pro-Hessian of H.*

The equations of a cubic and its Hessian being, as above respectively: $f \equiv x_1^3 + x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0$,

$$(1) \qquad H_f \equiv -m^2(x_1^3 + x_2^3 + x_3^3 - 6\mu \cdot x_1x_2x_3) = 0,$$

the Hessian of the Hessian, or second Hessian of the cubic, will have the following equation:

$$H_H(x) \equiv (-m^2)^3(-\mu^2)\{x_1^3 + x_2^3 + x_3^3 - 6\mu' \cdot x_1x_2x_3\} = 0, \quad \left(\mu' = \frac{1-2\mu^3}{6\mu^2}\right).$$

Expressed in terms of m, this is:

$$(2) \quad H_H(x) \equiv \frac{1}{216}\left\{\begin{matrix} 6m^2(1-2m^3)^2\,(x_1^3+x_2^3+x_3^3) \\ -(216m^6 - 2(1-2m^3)^3)x_1x_2x_3 \end{matrix}\right\} = 0.$$

From (1) and (2), eliminating the quantities: $(x_1^3 + x_2^3 + x_3^3)$ and $x_1x_2x_3$, we obtain an identical relation:

$$\begin{vmatrix} H_H(x) & -6m^2(1-2m^3)^2 & 216m^6 - 2(1-2m^3)^3 \\ f(x) & -216 & 6^4 m \\ H(x) & 216m^2 & -216(1-2m^3) \end{vmatrix} \equiv 0.$$

The importance of this relation lies in the fact that projection transforms these three cubics: $f = 0$, $H = 0$, and $H_H = 0$, into three others, each the Hessian of the one

preceding it. Hence the identical relation must have for coefficients functions of m which by linear transformation, (with any inflexional triangle as the triangle of reference), are unchanged except by some multiplicative constant. Expand the determinant, and remove the factor $(1 - 8m^3)$.

$$(3)\quad H_H(x) \equiv \frac{1}{27}(m + m^4)^2 \cdot f(x) + \frac{1}{108}(1 + 20m^3 - 8m^6) \cdot H(x).$$

The two functions of m in the right-hand member are the invariants usually denoted by S and T.

$$(4)\quad \begin{aligned} S &= m + m^4, \quad T = 1 + 20m^3 - 8m^6. \\ H_H &= \frac{1}{27} S^2 \cdot f + \frac{1}{108} T \cdot H. \end{aligned}$$

The equations (1) of cubic and Hessian differ only in the values of the parameters m and μ. Hence every cubic of the syzygetic sheaf (IV, § 6) has also an equation of the form:

$$(5)\quad x_1^3 + x_2^3 + x_3^3 - 6k x_1 x_2 x_3 = 0,$$

k denoting a parameter. We can use identity (4) to determine again the number of triangles which occur as degenerate cubics in this sheaf. A cubic consisting of 3 lines coincides with its Hessian, (for the latter meets it in the 9 base points and in its 3 double points, as shown in III, § 4). If then $H = 0$ denotes a triangle, $H_H = 0$ gives the same triangle, and if $f = 0$ denotes a proper cubic, since

$$H_H \equiv \frac{1}{27} S^2 f + \frac{1}{108} T \cdot H$$

we must have $S = 0$, i. e. $m - m^4 = 0$. This gives four values for m, hence as before we see that four triangles occur in the syzygetic sheaf, and now we note that each one is the Hessian curve of a proper cubic.

THEOREM. *Every cubic which has its invariant S equal to zero has for its Hessian curve one of its own inflexional triangles, and for its Cayleyan curve* (III, § 8) *the three vertices of that*

triangle. In every syzygetic sheaf containing proper cubics without double points there occur 4 *such special curves.*

If we set the other invariant equal to zero: $T = 0$, there are six values of m, 2 real and 4 imaginary; and by (4) we see that the curve coincides with its second Hessian.

THEOREM. *In a syzygetic sheaf there are in general six, two real and two pairs of conjugate imaginary cubics, each of which coincides with its second Hessian. These form three pairs, each member of a pair being the Hessian of the other.*

§ 2. The Problem of the inflexions algebraically formulated.

If the cubic is given by its equation, referred to an arbitrary triangle, how can the inflexional points and lines be found? We have seen that if the equation were in the form:

$$x_1^3 + x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0$$

the coordinates of the 9 inflexions could be written down at once (IV, § 4). The problem of finding the inflexions is therefore identical with that of reducing the equation to such a form, called the "Normal form of Hesse". For a real cubic without double point or cusp it was proven (IV, § 5) that the equation can be so reduced. Hence when the equation is given, it must first be discovered (by tests to be discussed later) that no double point exists; then there are inflexional triangles by whose aid the reduction can be effected, and the process of finding them would be as follows.

Let $f = 0$ denote the given cubic equation. Compute its Hessian $H_f = 0$. Construct the equation of a general cubic through the 9 inflexions: $f + \lambda \cdot H_f = \varphi = 0$. For this compute the Hessian H_φ and the second Hessian $H_\varphi^{(2)}$: the latter will be expressible as a linear combination of φ and H_φ (V, § 1): $H_\varphi^{(2)} = \frac{1}{27}(S_\varphi)^2 \cdot \varphi + \frac{1}{108} T_\varphi \cdot H_\varphi$. To determine $(S_\varphi)^2$ and T_φ it will be sufficient to compare two terms in φ and the corresponding terms in H_φ and $H_\varphi^{(2)}$. We use the

notation: S_φ, T_φ to indicate that S_φ and T_φ are related to the general cubic φ just as S and T (V, § 1) are related to f; the present equation is so far our only definition of these forms when φ is not reduced to normal form. Since φ contains λ to the first degree, H_φ must contain it to the third, $H_\varphi^{(2)}$ therefore to the ninth degree; hence in S_φ λ will occur to the fourth degree. Now for the curve: $H_\varphi = 0$ to degenerate into a set of three inflexional lines, it must coincide with its own Hessian (V, § 1), that is, if $\varphi = 0$ is a proper cubic, $H_\varphi^{(2)}$ must contain H_φ as a factor. According to the above identity, the condition for this is:

$$S\varphi = 0.$$

This is an equation of degree 4 in λ. If it were solved, the roots being denoted by λ_1, λ_2, λ_3, λ_4, the equations of the 4 triangles are

$$H_{(f+\lambda_1 H_f)} = 0, \qquad H_{(f+\lambda_3 H_f)} = 0,$$
$$H_{(f+\lambda_2 H_f)} = 0, \qquad H_{(f+\lambda_4 H_f)} = 0.$$

Each such Hessian is now the product of three linear factors, corresponding to inflexional lines. To discover these factors requires in each case the solution of two cubic equations. After factoring that one of the four, all of whose linear factors are real, it is sufficient to introduce those factors as three new variables, when the transformed equation will have necessarily the desired form, the normal form of Hesse. One reduction being found, the others are then already known. Cases requiring special investigation are those in which the equation: $S\varphi = 0$, has either more or fewer than four distinct roots.

§ 3. Invariants S and T of the Hessian. Pro-Hessians.

If the Hessian were written as

$$x_1^3 + x_2^3 + x_3^3 - 6\mu x_1 x_2 x_3,$$

its invariants corresponding to S and T could be found by substituting μ for m in the formula (§ 1):

$$S = m + m^4, \qquad T = 1 + 20m^3 - 8m^6.$$

Since the expression adopted (V, § 1) for the Hessian differs from the above by the factor $(-m^2)$, this factor must appear to the 4th power in the invariant S, to the 6th power in the invariant T, of the Hessian. Calling these S_H and T_H respectively, we have for their values:

(1)
$$S_H = m^8(\mu + \mu^4) = \frac{1}{6^4}\left[6^3 m^6(1 - 2m^3) + (1 - 2m^3)^4\right],$$

$$T_H \quad m^{12}(1 + 20\mu^3 - 8\mu^6)$$
$$= \frac{1}{6^6}\left[6^6 m^{12} + 20 \cdot 6^3 \cdot m^6(1 - 2m^3)^3 - 8(1 - 2m^3)^6\right].$$

Reducing these, we find them expressible rationally in S and T.

(2) $$S_H = \frac{1}{6^4}(T^2 - 48S^3), \quad T_H = \frac{1}{6^6}(8T(72S^3 - T^2)).$$

Eliminate S and find the equation for T in terms of S_H and T_H:

(3) $$12 \cdot 6^4 \cdot S_H \cdot T + 6^6 T_H = 4T^3.$$

This relation can be used, if the invariants of the Hessian are known, to find three values for the invariant T of its fundamental cubic. Recall now the relation (4) of (V, § 1):

(4) $$4S^2 \cdot f = 108H_H - T \cdot H.$$

If the equation: $H = 0$ of the Hessian were given, we have here a means of finding the fundamental cubic, unless its invariant S is equal to zero. If $S = 0$, from (2) we see that we must have a relation:

(5) $$T_H^2 - 64S_H^3 = 0$$

between the invariants of the given Hessian. Whenever this is not satisfied, three values of T can be determined from the cubic (3), the Hessian of the Hessian can be computed, then the right-hand member of (4) equated to zero will be the equation of a fundamental cubic for the given Hessian.* This proves the following theorem.

THEOREM. *Every given cubic whose invariants do not satisfy the relation:* $T^2 - 64S^3 = 0$ *has three other cubics as pro-Hessians, and no two of the three can coincide. When the given cubic is real, if* $T^2 - 64S^3$ *is negative, the three pro-Hessians are all real; if* $T^2 - 64S^3$ *is positive, one is real and two are imaginary.*

The proof of the latter part of this theorem comes from considering the discriminant of the cubic equation (3) in T. If the roots of (3) are real and unequal, two equal, or two imaginary, the pro-Hessians: $f = 0$, given by formula (4) will be real and different, two coincident, or two imaginary. Real cubics are thus divided into two classes by the nature of their pro-Hessians, and *the compound invariant*: $(T^2 - 64S^3)$ *is called the Discriminant of the cubic.*

§ 4. Cross-ratio of four tangents from a point of the Cubic. Conditions for a double-point and for a cusp.

From any ordinary point (x) of a cubic which has neither doublepoint nor cusp four tangents can be drawn to the curve; their equation is (III, § 3):

$$(1) \qquad 9(ax)(ay)^2 \cdot (bx)(by)^2 - 12(ax)^2(ay) \cdot (by)^3 = 0.$$

The cross-ratio of the four tangents is identical with that of the four points in which they meet an arbitrary line, as the line: $y_3 = 0$. If in equation (1) we set $y_3 = 0$, it becomes a quartic equation in the ratio $y_1 : y_2$, having as coefficients certain functions of the coordinates (x). But the cross-ratio of the four roots $(y_1 : y_2)$ of that equation depends only upon

* Note that it is not stated that when $T^2 - 64S^3 = 0$, a pro-Hessian cannot be found. That is not true. See article by P. Gordan, Trans. of the Amer. Math. Society, Vol. 1 (1900), pp. 402-413.

the absolute invariant i^3/j^2 of that quartic equation,* that is, upon a rational fractional function of the coordinates of the point (x) on the cubic, $\varphi(x)/\psi(x)$. Hence when this cross ratio is equal to any constant k, the value of $\varphi(x)/\psi(x)$ must assume a corresponding value K i. e. (x) must satisfy the equation: $\varphi(x) - K \cdot \psi(x) = 0$. When we give the cross-ratio k the value $+1$, two points must coincide. Let K_1 be the corresponding value of K, then the equation

$$\varphi(x) - K_1 \cdot \psi(x) = 0 \tag{2}$$

must denote a curve cutting the cubic: $f(x) = 0$, in points from which double tangents can be drawn to the cubic. The inflexional points cannot count as such points, because from them three tangents can be drawn which do not coincide with the inflexional tangent; and since no line can be tangent to the cubic in two distinct points, the curve (2) does not intersect the cubic. Hence the order of the curve (2) cannot be different from zero, i. e. the ratio $\varphi(x)/\psi(x)$ is constant at all points of the cubic.

To prove this by explicit calculation, use the normal form:

$$f(x) = (ax)^3 = x_1^3 + x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0. \tag{3}$$

From this, computing the terms of (1) and setting $y^3 = 0$, we have for the biquadratic in $y_1 : y_2$ the following:

$$\begin{aligned} &y_1^4(x_1^2 - 8mx_2x_3) + 4y_1^3y_2(x_2^2 + mx_1x_3) - 6y_1^2y_2^2(2m^2x_3^2 + x_1x_2) \\ &\quad + 4y_1y_2^3(x_1^2 + mx_2x_3) + y_2^4(x_2^2 - 8mx_1x_3) = 0. \end{aligned} \tag{4}$$

For brevity write the equation thus:

$$a_{40}y_1^4 + 4a_{31}y_1^3y_2 + 6a_{22}y_1^2y_2^2 + 4a_{13}y_1y_2^3 + a_{04}y_2^4 = 0.$$

* See Burnside and Panton: Theory of Equations (1899), Vol. 1. Ch. 6, Example 16; p. 148. Also Clebsch: Binare Formen (1872), p. 169, § 50; and Andoyer: Théorie des formes (1900), Vol. 1, p. 100, § 97.

The two invariants of this binary form, i and j, are thus defined:*

$$i = a_{40}a_{04} - 4a_{31}a_{13} + 3a_{22}^2,$$

$$j = \begin{vmatrix} a_{40} & a_{31} & a_{22} \\ a_{31} & a_{22} & a_{13} \\ a_{22} & a_{13} & a_{04} \end{vmatrix}.$$

Introducing the values of the a_{ik} from equation (4), we find that x_1 and x_2 can be eliminated together by the aid of (3).

$$i = 12(m + m^4)x_3^4 = 12x_3^4 \cdot S,$$
$$j = x_3^6(1 + 20m^3 - 8m^6) = x_3^6 \cdot T.$$

The cross-ratio k is any root of the equation:†

$$(5) \qquad \frac{(k^2 - k + 1)^3}{(k+1)^2(k-2)^2(2k-1)^2} = \frac{i^3}{108j^2} = \frac{16S^3}{T^2}.$$

Now S and T are invariants, and the ratio S^3/T^2 does not depend upon the coordinates of any particular point on the cubic. Hence the equation (5), from which k is to be found, is the same for all points of the curve. This confirms the proof of the theorem following.

SALMON'S THEOREM. *The cross-ratio of the four rays drawn tangent to a non-singular cubic from any point on the curve is a constant number, the same for all points upon the cubic.*

If two non-singular cubics have this constant cross-ratio equal, they are projective to each other. For at an inflexional point A of the one, the inflexional tangent AT and the other three tangents: AB_1, AB_2, AB_3, are projective to four similar lines: $A'T'$, $A'B_1'$, $A'B_2'$, $A'B_3'$ of the other. It is always possible to project four points $AB_1B_2B_3$ upon four others, $A'B_1'B_2'B_3'$, hence the fourth line AT will fall upon $A'T'$,

* ANDOYER'S Théorie des formes, (1900), Vol. 1, pp. 97 seqq. § 92.
† Ibidem, §§ 96, 97.

and the two cubics have in common 3 ordinary points with their tangents and an inflexion with its tangent, the equivalent of 9 points. But since the 3 ordinary points lie on a line (the harmonic polar of A or A' (IV, § 1)), any one additional point C of the first curve can be brought to coincide with C', the point in which AC now meets the second curve. Having then 10 common points, the two cubics must coincide (II, § 1).

THEOREM. *Not only must two projective cubics have the same cross-ratio of the four tangents drawn from any one of their points, but conversely any two cubics characterized by equal cross-ratios are projective. Hence all cubics whose equations give to the invariant ratio $S^3 : T^2$ the same value are projective to one another.*

DEFINITION. *The ratio $S^3 : T^2$ is called the absolute invariant of a cubic with respect to projective transformations.*

When the cubic has a double point, the number of proper tangents through every point is diminished by 2; by 3, if it has a cusp (III, § 3). In case of a double point occuring on the curve, therefore, two of the four tangents here considered must coincide in the line from (x) to the double point; the cross-ratio k must then have a value: $k = 1$. In case of a cusp, three tangents have come to coincide in the line joining (x) to the cusp, whence k is indeterminate. By equation (5), the value: $k = 1$ gives the condition: $T^2 - 64S^3 = 0$; and for k indeterminate we should have $T = S = 0$.

THEOREM. *The proper cubic curve has a double point if its equation gives for S and T values, different from zero, satisfying the condition: $T^2 - 64S^3 = 0$. The proper cubic has a cusp, when both $T = 0$ and $S = 0$, and not otherwise.*

For the argument is convertible; four rays have a cross-ratio: 0, 1, or ∞ only when two coincide; and an indeterminate cross-ratio only when three coincide.

If the tangents form a *harmonic* set, k has the value -1, 2, or $\frac{1}{2}$, therefore $T = 0$. The set is called *equianharmonic* when $k^3 = 1$, $k \neq 1$, therefore $k^2 + k + 1 = 0$; for this value

we find $S = 0$. These names are often attached to the curves characterized by these particular cross-ratios.

DEFINITION: *Harmonic cubics are those whose tangents from any point of the curve form a harmonic pencil; and such curves have their invariant* $T = 0$. *Equianharmonic cubics have for their characteristic cross-ratio an imaginary cube root of unity, and their invariant* $S = 0$, *while their Hessians are degenerate* (V, § 1).

For the division of cubics without double points into classes (V, § 3) there is a second and more simple geometrical reason. If a cubic has only the odd branch (I, § 3), from any point on that branch only two real tangents can be drawn. (See Note II of Appendix). If the cubic has two branches, on the other hand, from points on the even branch no real tangents can be drawn, while from points on the odd branch four real tangents radiate, two to the odd branch, two to the even branch. In equation (4) the case of two real and two imaginary roots is distinguished from those of either four real or four imaginary roots by the sign of its discriminant $(i^3 - 27j^2)$. This is negative in the former case, positive in the latter. But by equation (5), from $i^3 - 27j^2 \gtrless 0$ it follows that

$$T^2 - 64S^3 \lesseqgtr 0.$$

This together with § 3 of this chapter gives the following theorem.

THEOREM. *When the discriminant of a real cubic is positive,* $T^2 - 64S^3 > 0$, *the curve is unipartite, and is the Hessian of one real and two imaginary cubics. If the discriminant is negative,* $T^2 - 64S^3 < 0$, *the curve is bipartite and is the Hessian of three real cubics.*

From the values of S_H and T_H, the two invariants of a Hessian, we can determine whether the cubic and its Hessian are of the same class or not. Those values are (§ 3)

$$S_H = \frac{1}{6^4}(T^2 - 48S^3), \quad T_H = \frac{1}{6^6}(8T(72S^3 - T^2)).$$

Substituting these values in the discriminant of the Hessian we find

$$T_H^2 - 64S_H^3 = \frac{2^6}{3^9}(-S^6)(T^2 - 64S^3).$$

Since S is a real quantity, $-S^6$ is negative, hence the cubic and its Hessian have discriminants of unlike sign. This gives the theorem:

THEOREM. *A bipartite real cubic curve has a unipartite Hessian, and vice versa. The Hessian can have a node only when* (1) *either the cubic itself has a node or* (2) *the cubic is equianharmonic* (*i. e. when* $S = 0$).

As a corollary we can infer that a real cubic cannot be projective to its Hessian unless both have nodes; for real collineation or projection cannot relate a unipartite to a bipartite curve.

For a cubic that has a real node, the lines which intersect the curve triply at the node may be axes $x_1 = 0$ and $x_2 = 0$; take the tangent in the real inflexion for $x_3 = 0$. Then the curve and its Hessian have the equations:

$$(x_1 + x_2)^3 - 6ax_1x_2x_3 = 0,$$
$$(x_1 + x_2)(x_1 - x_2)^2 + 2ax_1x_2x_3 = 0.$$

These have nodes at the same point, but are convex in opposite directions toward the axes at that point. Also, common points all lie on the 3 lines through the nodes

$$(x_1 + x_2)[(x_1 + x_2)^2 + 3(x_1 - x_2)^2] = 0.$$

Therefore there are 3 distinct points of intersection, and no more, outside the node, — one real, two imaginary. *A cubic curve with a real node has one real, two imaginary inflexions, the node is said to absorb the other six.* A curve with a cusp is discussed in similar manner, and the cusp is found to "absorb" eight inflexions.

Chapter VI.

Pairs of corresponding points on the cubic.

§ 1. Tangents at corresponding points meet on the curve.

If (x) and (y) denote corresponding points on a cubic whose equation is $(Hz)^3 = 0$, while the equation of the pro-Hessian which mediates the correspondence is $(az)^3 = 0$, then the tangents at (x) and (y) respectively are given, (III, § 14) by:

$$(1) \qquad (ay)^2(az) = 0, \quad (ax)^2(az) = 0.$$

The relation between (x) and (y) is given by the 3 equations (III, § 4):

$$(2) \quad a_1(ax)(ay) = 0, \quad a_2(ax)(ay) = 0, \quad a_3(ax)(ay) = 0.$$

Or for any arbitrary point (z): $(ax)(ay)(az) = 0$. Denote by (z) the intersection of the two tangents (1). The conic polar of (z) contains not only (x) and (y), but all points of the line (x, y); for if $t_i = x_i + \lambda y_i$, $(i = 1, 2, 3)$, we have:

$$(3) \quad (az)(at)^2 = (az)(ax)^2 + 2\lambda(az)(ax)(ay) + \lambda^2(az)(ay)^2 = 0,$$

because by equations (1) and (2) all three terms of the second member in (3) vanish independently. Therefore the conic polar of (z) must consist of the line (x, y) and another line, and (z) therefore must lie on the Hessian, i. e. on the cubic $(Hz)^3 = 0$. As (x, y) is one part of the conic polar of (z), the point corresponding to (z) must lie on (x, y), namely at its third intersection with the cubic $(Hz)^3 = 0$.

Theorem. *The line joining two corresponding points on a cubic considered as a Hessian forms part of the conic polar of a third point of the cubic with respect to the pro-Hessian, and in that third point, or pole, the tangents at the first two*

points intersect. The point of the cubic corresponding to this intersection of the tangents is the third intersection of the cubic with the secant through the first two points.

The Cayleyan was defined (III, § 8) as the envelope of degenerate conic polars. The theorem just proven enables us to add: *the Cayleyan is also the envelope of lines joining pairs of corresponding points on a Hessian.* Is then every constituent line in a degenerate conic polar a join line of two corresponding points on the Hessian?

§ 2. Every tangent of the Cayleyan joins two corresponding points of the Hessian.

Assume two corresponding points A, B on a proper cubic considered as a Hessian. Call the cubic H, its pro-Hessian F, and its Cayleyan K. The conic polar of B with respect to F consists of two lines, $AC_1\,C_2$ and $AD_1\,D_2$, where C_1, C_2, D_1, D_2 denote points upon H. It is to be proven that C_1 and C_2 are corresponding points, as also D_1 and D_2. Let C_1' be the corresponding point to C_1. Through C_1 will pass three tangents to the Cayleyan K, namely two constituting the conic polar of C_1', (dotted lines in Fig. 20), and the join line of C_1 and C_1' (III, § 8 and VI, § 1). With which of these three does $AC_1\,C_2$ coincide? With neither of the first two, unless the pole C_1' coincides with B,* the one point having $AC_1\,C_2$ for part of its conic polar; were this so, it would require C_1 to coincide with A, and the line AC_1' to be tangent in A, i. e. to be the linear polar of B with respect to F. But the linear polar cannot coincide with part of the conic polar, unless the latter is a doubly counting line and counts as part of a degenerate Hessian; whereas H is taken as a proper cubic. Since no point contains more than three

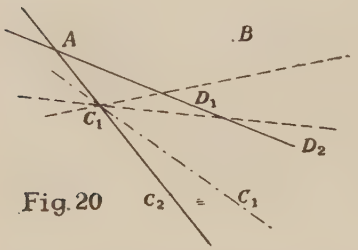


Fig. 20

* If B and C_1' both had $C_1\,C_2$ as part of their conic polars, every point on $C_1\,C_2$ would have the line BC_1' for its linear polar, whence $C_1\,C_2$ would lie wholly in the Hessian H.

tangents to the Cayleyan, and $C_1 C_2$ does not coincide with either of the first two, it must coincide with the third, namely with $C_1 C_1'$; i. e. C_1' must be the point C_2. q. e. d. Similarly the point corresponding to D_1 is D_2. The converse of a theorem in the preceding section is therefore true, as follows.

THEOREM. *Every tangent to the Cayleyan, that is, every line constituting part of a degenerate conic polar with respect to a pro-Hessian, connects two corresponding points on the cubic.* This joined with the latter part of the theorem in § 1 gives an interesting corollary: *the four points, different from the double-point, in which a degenerate conic polar meets the Hessian, have a common tangential point, namely the pole of this degenerate conic.*

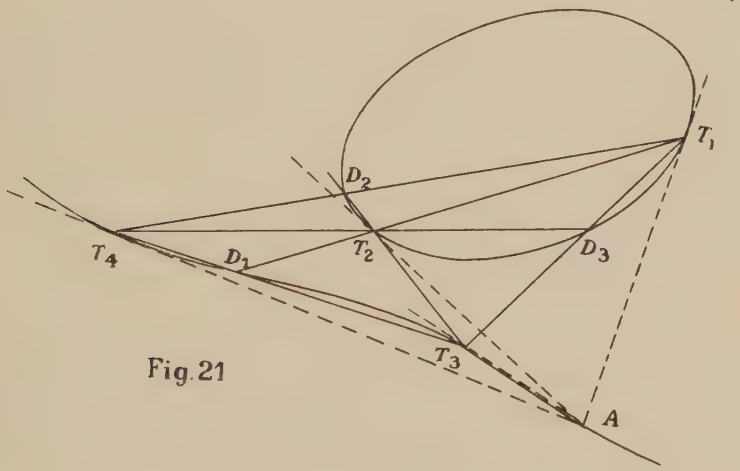


Fig. 21

Another important corollary is obtained by altering the terms, and recalling the prime property of conic polars (III, § 3) as containing the contacts of tangents from the pole. *The conic polar of a point on a proper cubic and the conic polar of the same point with respect to a pro-Hessian* of the cubic intersect in four points of the cubic, namely in the contacts of the four tangents drawn from that point to the cubic. The*

* Therefore the conic polars of the same point with respect to *all cubics* of the syzygetic sheaf pass through the same four points.

three degenerate conics through the four points are the conic polars of the first point with respect to the three pro-Hessians of the cubic.

This last proposition may be stated in another form.

THEOREM. *If from any point of a proper cubic the four tangents are drawn to the curve, the four points of contact are vertices of a complete quadrangle whose three diagonal points are also points of the cubic.*

In Fig. 21, if AT_1, AT_2, AT_3, AT_4 are tangents to the cubic, the diagonal points D_1, D_2, D_3, are on the curve, and each is corresponding point to A with respect to some one of the 3 pro-Hessians.

§ 3. Degenerate Poloconics.

If a line forms part of a degenerate conic polar, its poloconic becomes degenerate. Let a point D lie on the Hessian, its conic polar consisting of two lines a, b, cutting the Hessian

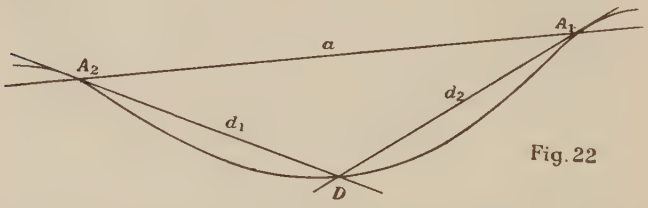


Fig. 22

respectively in points $CA_1\,A_2$, $CB_1\,B_2$. What can be inferred as to the poloconic of the line a? All points of a are in the conic polar of D, accordingly all linear polars of such points will pass through D. Therefore the poloconic as the envelope of these linear polars must reduce to the point D. On the other hand, as the locus of poles of conic polars tangent to a, it can only reduce to two lines d_1, d_2, and these must pass through D. Further, as every poloconic is tangent to the Hessian in 3 points, D will count as an improper contact, while the other two must be proper contacts of d_1 and d_2 with the Hessian. Where are these contacts? Call them A_3 and A_4. Since A_3 and A_4 are on the poloconic of a, their conic polars must touch a; and since A_3 and A_4 are on the

Hessian, these conic polars are degenerate, and only their double points on the Hessian can count as contacts. As the double points lie on a and the Hessian, they are the points A_1 and A_2. Therefore the poles A_3 and A_4 of those conic polars are A_2 and A_1, for these points correspond reciprocally. The lines d_1 and d_2 are therefore no other than the tangents DA_2 and DA_1. (See Fig. 22). The theorem thus proven is as follows.

THEOREM. *Every line joining two corresponding points of the non-singular cubic has with respect to the pro-Hessian a degenerate poloconic consisting of the tangents in those two points.* The converse is readily proved. *Every degenerate poloconic of a cubic whose Hessian is non-singular consists of two tangents in corresponding points on the Hessian.* To display a certain duality we may state it in other words: *every tangent to the Cayleyan has as poloconic two tangents to the Hessian, and every degenerate poloconic has its double point on the Hessian and a chord of contact touching the Cayleyan.*

RESUMÉ. As a cubic has three pro-Hessians, so it is associated with three Cayleyans. Four tangents from any point of the cubic have contacts which fall into pairs in three ways, corresponding to the three pro-Hessians, and the three pairs of lines joining them two and two are pairs of corresponding tangents to the respective Cayleyans; and we have now recognized in the three times two pairs of tangents, in which the four can be arranged, degenerate poloconics of the three systems.

§ 4. Corresponding points on a cubic are conjugate points for all conics of a net.

Considering a non-singular cubic: $(Hx)^3$ 0 as the Hessian of another: $(ax)^3 = 0$, two corresponding points of the former are defined by the relations:*

$$(1) \qquad a_i\,(ax)(ay) = 0, \qquad i = 1,\ 2,\ 3.$$

* See Chapter III, § 4, or § 1 of this chapter.

Using as multipliers for these 3 equations the coordinates t_1, t_2, t_3 of an arbitrary point, and adding, we have

$$(at)(ax)(ay) \equiv 0. \tag{2}$$

This may be read as follows: (y) lies on the polar of (x) with respect to the conic in coordinates (z):

$$(at)(az)^2 = 0. \tag{3}$$

In other words, (x) and (y) are *conjugate points* with respect to the conic (3), and as the three t's are arbitrary, the conic (3) is any conic of a doubly infinite linear system. Such a system is called a *net* of conics. With respect to the cubic: $(ax)^3 = 0$, the conics (3) are first polars, and we may say: (x) *and* (y), *corresponding points on the Hessian, are conjugate with respect to the net of conic polars.* The term "conjugate points" on a cubic, is more generally used and more significant than the equivalent, "corresponding points". The most important property of pairs of conjugate points is involved in a theorem upon conics, due to Hesse.* That theorem, usually made to depend upon a property of involutions or of conjugate triangles, can be proven quite as readily and directly by the aid of abbreviated notation.

§ 5. Hesse's lemma upon two pairs of points conjugate with respect to the same conic.

If two pairs of points AA', BB' are conjugate pairs with respect to a conic, then the two pairs of lines AB, $A'B'$ and AB', $A'B$ have their points of intersection C, C' conjugate with respect to the same conic.

Proof. Denote the points by coordinates as follows (Fig. 23): A and A' by $(x_1 x_2 x_3)$ and $(\xi_1 \xi_2 \xi_3)$, B and B' by $(y_1 y_2 y_3)$ and $(\eta_1 \eta_2 \eta_3)$, C and C' by $(s_1 s_2 s_3)$ and $(\sigma_1 \sigma_2 \sigma_3)$. If the equation of the conic is: $(az)^2 = 0$ the given conditions are:

* Crelle's Journal für reine und angewandte Mathematik, Vol. 20 (1840) p. 301. See also Reye: Geometrie der Lage (3rd edition) Vol. 1, p. 220, section 166, or Holgate's translation, p. 215, § 22 of Appendix. An interesting geometrical proof, with references, is given by Hofmann in the Journal für r. u. a. Math., Vol. 102 (1888) p. 175, seqq.

$$(1) \quad (ax)(a\xi) = 0, \qquad (ay)(a\eta) = 0.$$

$$(2) \quad (xys) = 0, \quad (\xi\eta s) = 0; \qquad (x\eta\sigma) = 0, \quad (\xi y\sigma) = 0.$$

The theorem to be proven is: $(as)(a\sigma) = 0$.

Express by the aid of equations (2) the coordinates of A and A' in terms of the other four points. Let $(ys)_1$ denote the determinant:

$$(y_2 s_3 - y_3 s_2) = (ys)_1, \text{ etc.}$$

Then we find:

$$\begin{aligned} \varrho x_1 &= (ys)_2(\eta\sigma)_3 - (ys)_3(\eta\sigma)_2 \\ (3) \quad \varrho x_2 &= (ys)_3(\eta\sigma)_1 - (ys)_1(\eta\sigma)_3 \quad (\varrho \text{ a factor of proportionality}). \\ \varrho x &= (ys)_1(\eta\sigma)_2 - (ys)_2(\eta\sigma)_1 \end{aligned}$$

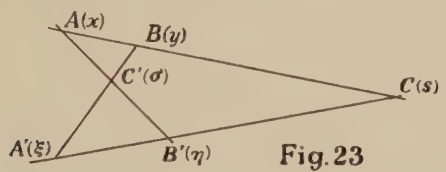


Fig. 23

Similarly the coordinates $\xi_1 \xi_2 \xi_3$ are expressed. Substitute in (1):

$$(4) \quad \begin{vmatrix} a_1 & (ys)_1 & (\eta\sigma)_1 \\ a_2 & (ys)_2 & (\eta\sigma)_2 \\ a_3 & (ys)_3 & (\eta\sigma)_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & (\eta s)_1 & (y\sigma)_1 \\ a_2 & (\eta s)_2 & (y\sigma)_2 \\ a_3 & (\eta s)_3 & (y\sigma)_3 \end{vmatrix} = 0.$$

Multiply these factor-determinants respectively by

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ s_1 & s_2 & s_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} \text{ and } \begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ s_1 & s_2 & s_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

The result is:

$$0 = \begin{vmatrix} (ay) & (as) & (a\eta) \\ 0 & 0 & (\eta ys) \\ (y\eta\sigma) & (s\eta\sigma) & 0 \end{vmatrix} \cdot \begin{vmatrix} (a\eta) & (as) & (ay) \\ 0 & 0 & (y\eta s) \\ (\eta y\sigma) & (sy\sigma) & 0 \end{vmatrix}.$$

Remove the inserted factors $(ys\eta)$ and (ηsy), there remains:

$$0 = \{(ay)(s\eta\sigma) - (as)(y\eta\sigma)\} \cdot \{(a\eta)(sy\sigma) - (as)(\eta y\sigma)\}.$$

Multiply, and drop the term containing $(ay)(a\eta)$, which vanishes by equation (1).

$$(5) \quad 0 = \{(as)^2(y\eta\sigma) + (as)(a\eta)(sy\sigma) - (as)(ay)(s\eta\sigma)\} \cdot (\eta y\sigma)$$

As the points $BB'C'$ are not collinear, $(\eta y\sigma) \neq 0$ and may be omitted. Factoring out (as), the three terms may be thought of as the expansion of the following determinant.*

$$0 \quad \begin{vmatrix} (as) & s_1 & s_2 & s_3 \\ (ay) & y_1 & y_2 & y_3 \\ (a\eta) & \eta_1 & \eta_2 & \eta_3 \\ 0 & \sigma_1 & \sigma_2 & \sigma_3 \end{vmatrix} (as).$$

Add to the first column the second multiplied by $-a_1$, the third by $-a_2$ and the fourth by $-a_3$.

$$(6) \quad 0 = (as) \begin{vmatrix} 0 & s_1 & s_2 & s_3 \\ 0 & y_1 & y_2 & y_3 \\ 0 & \eta_1 & \eta_2 & \eta_3 \\ -(a\sigma) & \sigma_1 & \sigma_2 & \sigma_3 \end{vmatrix} = (as)(a\sigma)(sy\eta).$$

Or since the points B, B', C are not collinear and hence $(sy\eta) \neq 0$,

$$(as)(a\sigma) = 0. \qquad \text{q. e. d.}$$

A shorter statement of the lemma is this: *in a complete quadrilateral, if two pairs of opposite vertices are conjugate with respect to a conic, the third pair of vertices also must be conjugate with respect to that conic.*

Note that the construction fails if any three of the first four points are collinear.

* The reader not familiar with the algebra of invariants should note carefully this step, employing a formal identity of frequent use in that theory.

§ 6. Schroeter's linear construction of points on a cubic from three pairs of conjugate points.

Conjugate points on a Hessian are conjugate with respect to all conics of a net. If they are conjugate with respect to any three which are linearly independent, they are so also with respect to all conics of the net in which those three lie. By the lemma proven above, it is possible to derive a third conjugate pair from two given pairs of points; there however the process stops. From A, A' and B, B' can be derived C, C', but the same method applied to A, A' and C, C' yields only B, B'. But if three pairs are given, the process will generally be interminable.* Taking the 3 pairs two and two, 3 new pairs can be found. Combining these with one another, and each with that one of the first 3 from which it was not derived, six new pairs are found. This continued derivation of new pairs is remarkable as being effected by the aid of the ruler alone; it is the simplest possible method of geometrical derivation. So far it has not been shown that the derived points all lie on a cubic unless the first three pairs were expressly given as conjugate pairs on a determinate cubic. This partial result may be stated as a theorem.

THEOREM. *Regarding a given cubic as the Hessian of another, if there are given three pairs of conjugate points lying upon it, no three points lying in a right line, then by the aid of right lines alone there can be found in special cases a finite number, in the general case an indefinite number of other pairs of conjugate points on the same cubic.*

In this it has been assumed, not that all real points of the cubic were known, but only that the specified points were known, and were certified as conjugate pairs. Suppose however that the curve, i. e. its real branches, were completely

* The process might terminate, owing to special relations among the given points, as HURWITZ has shown (Journal für r. u. a. Math., Vol. 107, pp. 141–7). For example, from 3 of the inflexions and their conjugate points could be derived only the remaining six inflexions and their six conjugate points, for one of each new pair would necessarily lie on a line with two inflexions, and so must be itself an inflexion.

drawn, and that this problem were proposed: to determine the pairs of conjugate points upon these real branches. In this case also Hesse's theorem would accomplish the solution if one pair of conjugate points were known. For if A_1 and A_2 are the conjugate points given, B_1 and B_2 any other conjugate pair not yet found, then (A_1, B_1) and (A_2, B_2) must intersect at some point C_1 on the curve. Conversely if we draw through any point C_1 of the cubic the lines (A_1, C_1), (A_2, C_1), they must cut the curve further in two points B_1 and B_2 mutually conjugate. (Draw (A_1C_1) cutting the curve in B_1, then the line joining A_2 to the conjugate of B_1 must also contain C_1, therefore B_2 is conjugate to B_1). Think of A_1 and A_2 as centers about which turn two secants constrained to pass through a variable point C_1. Then let C_1 describe the real branches of the cubic; the third intersections of the two secants with the curve will evidently mark upon it constantly conjugate pairs of points.

THEOREM. *Further it can be shown that three perfectly arbitrary pairs of points, provided no three are in a line, may be used as a basis for Schroeter's construction, and the derivative points will all lie on a cubic containing the first three pairs.*

PROOF. Designate the given points by X and X', Y and Y', Z and Z', with coordinates (x), (x'), etc. respectively. Determine 3 linearly independent conics such that these three point-pairs are conjugate with respect to each one. then every derivative pair will also be conjugate with respect to the same 3 conics, and therefore with respect to every conic of the net containing the 3. For the sake of definiteness take for equation of any conic of the net:

$$(1) \qquad (at)^2 = \lambda_1 t_1^2 + \lambda_2 t_2^2 + \lambda_3 t_3^2 + 2(l_1 t_2 t_3 + l_2 t_3 t_1 + l_3 t_1 t_2) \qquad 0.$$

Impose the conditions of conjugateness in terms of the coordinates of the six given points:

$$(2) \qquad (ax)(ax') = 0, \quad (ay)(ay') = 0, \quad (az)(az') = 0.$$

From these 3 conditions, l_1, l_2 and l_3 can be determined as linear functions of λ_1, λ_2, λ_3, three arbitrary parameters. The equation (1) then containing linearly three parameters, represents a net of conics, to which net (by the preceding section) all derivative pairs of points will be conjugate. That these points all lie on a fixed cubic is now to be proven.

For the values $\lambda_1 : \lambda_2 : \lambda_3 = 1 : 0 : 0$, let $(at)^2$ become in particular $(a't)^2$. So for $(\lambda_1 \lambda_2 \lambda_3) = (010)$ or (001), call the particular forms $(a''t)^2$ and $(a'''t)^2$. If now (ξ) and (η) denote any pair of conjugate points, the 3 equations are satisfied:

$$(3) \qquad (a'\xi)(a'\eta) = 0, \quad (a''\xi)(a''\eta) = 0, \quad (a'''\xi)(a'''\eta) = 0.$$

Eliminating the coordinates (η) from these equations, just as in III, § 4, we have a cubic equation for (ξ).

$$(4) \qquad \begin{vmatrix} a_1'(a'\xi) & a_2'(a'\xi) & a_3'(a'\xi) \\ a_1''(a''\xi) & a_2''(a''\xi) & a_3''(a''\xi) \\ a_1'''(a'''\xi) & a_2'''(a'''\xi) & a_3'''(a'''\xi) \end{vmatrix} = 0.$$

By eliminating (ξ) the same equation would have been found for (η). This is the equation of a cubic containing both points of every pair conjugate with respect to the net. Hence the 6 given points and all derivative points lie on a determinate cubic.

By Schroeter's construction we have seen that 3 pairs of points arbitrarily chosen lead to an indefinitely great number of pairs, all conjugate with respect to the conics of a net. Conjugate pairs on a Hessian had this character. Is then the cubic of equation (4) the Hessian of another whose conic polars are the net in equation (1)? If this were not so, a cubic might contain, beside the 3 systems of conjugate pairs of points due to its 3 pro-Hessians, other systems due to nets of conics which are not the conic polars of any cubic. That this does not occur can be shown now, in advance of the direct demonstration of *Hermite's* theorem.

§ 7. A cubic cannot have more than three systems of conjugate points.

From Hesse's lemma it follows (VI, § 5) that if the points of a cubic are conjugate in pairs with respect to a net of conics, tangents to the curve in the points of any pair must intersect upon the curve. For the second pair may be taken as near as we please to the first, and in the limit one pair of lines joining the one pair of points to the other will become tangents, while the diagonal point at their intersection remains upon the cubic. Now the number of tangents meeting in any point of the cubic is never more than 4, hence to any given point there are not more than 3 having the same tangential point, i. e. not more than 3 conjugate points in as many different systems, and those 3 relate to nets of conics which are conic polars of the 3 pro-Hessians (VI, § 3, Resumé). Hence *every net of conics whose pairs of conjugate points lie all on a given non-singular cubic must constitute the system of conic polars of one of the* 3 *pro-Hessians of that cubic.* (Compare the last theorem in III, § 6 and that in III, § 14).

§ 8. The conics of a net are conic polars of a cubic.

The conic polars, with respect to a cubic, of all points in the plane form a net of conics. Is the converse true? The foregoing sections seem to prove it, since the locus of conjugate points is a cubic (§ 6), and can contain (§ 7) only 3 systems of conjugate points; while it has 3 pro-Hessians, one of which therefore must have the given net for its conic polars. The theorem is however so fundamental that we shall give a direct proof.

If the conics of a given net:

$$\lambda_1 (ax)^2 + \lambda_2 (bx)^2 + \lambda_3 (cx)^2 = 0$$

are conic polars of the points (y) of a plane with respect to a cubic: $f(x) = 0$, then the parameters (λ) are linear

functions of the coordinates of the pole (y). Set therefore:

$$\begin{aligned}(1)\qquad &\lambda_1(ax)^2+\lambda_2(bx)^2+\lambda_3(cx)^2\\ &\equiv \Phi(y,x)\quad\left(y_1\frac{\partial}{\partial x_1}+y_2\frac{\partial}{\partial x_2}+y_3\frac{\partial}{\partial x_3}\right)\cdot f(x),\end{aligned}$$

$$\begin{aligned}(2)\qquad &\lambda_1=(\alpha y)\equiv\alpha_1y_1+\alpha_2y_2+\alpha_3y_3,\\ &\lambda_2=(\beta y),\ \lambda_3=(\gamma y).\end{aligned}$$

The identity (1) gives for partial derivatives of $f(x)$:

$$(3)\qquad \frac{\partial f}{\partial x_i}\equiv\alpha_i(ax)^2+\beta_i(bx)^2+\gamma_i(cx)^2,\qquad (i=1,2,3).$$

For these to be derivatives of the same function $f(x)$, the necessary and sufficient conditions are of the form:

$$(4)\qquad \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_k}\right)\equiv\frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_i}\right)\qquad (i,k=1,2,3).$$

There result the equations following, equivalent to 8 conditions.

$$\begin{aligned}(5)\qquad &\alpha_1a_{23}+\beta_1b_{23}+\gamma_1c_{23}\\ &\quad=\alpha_2a_{13}+\beta_2b_{13}+\gamma_2c_{13}=\alpha_3a_{12}+\beta_3b_{12}+\gamma_3c_{12},\\ &\alpha_ia_{kk}+\beta_ib_{kk}+\gamma_ic_{kk}\\ &\quad=\alpha_ka_{ik}+\beta_kb_{ik}+\gamma_kc_{ik},\qquad (i,k=1,2,3).\end{aligned}$$

In these eight equations (5) the nine parameters (α), (β), (γ) are contained linearly: the solution is therefore neither impossible nor indeterminate unless at least all the 8-rowed determinants of their matrix become zeros. By an example it can be shown that those determinants are not identically zero for every net of conics. Thereupon we shall find what geometric conditions are necessary (but not sufficient) to render the solution indeterminate.

If there are in the net two conics not in contact, assume those for two fundamental conics, and denote them by:

$(bx)^2 = 0$ and $(cx)^2 = 0$. Referred to their unique self-conjugate triangle they have the equations:

$$(bx)^2 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 = 0,$$
$$(cx)^2 = c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 = 0.$$

As these are not in contact, it is certain that

(6) $\quad b_{22} : c_{22} \neq b_{33} : c_{33},\ b_{33} : c_{33} \neq b_{11} : c_{11},$ and $b_{11} : c_{11} \neq b_{22} : c_{22}.$

Selecting then as a third fundamental conic one whose equation contains as few terms as may be, we have:

$$(ax)^2 = a_{11}x_1^2 + 2(a_{23}x_2x_3 + a_{13}x_1x_3 + a_{12}x_1x_2).$$

Thus the equations of condition are simplified, and their matrix is the following, whose determinants do not vanish identically.

(7)
$$\begin{array}{cccccccccc}
a_{23} & -a_{13} & & & & & & & & \\
a_{23} & & -a_{12} & & & & & & & \\
 & -a_{12} & & & b_{22} & c_{22} & & & & \\
 & & -a_{13} & & b_{33} & c_{33} & & & & \\
-a_{12} & a_{11} & & & & & b_{11} & c_{11} & & \\
 & & -a_{23} & & & & b_{33} & c_{33} & & \\
-a_{13} & & a_{11} & & & & & & b_{11} & c_{11} \\
 & -a_{28} & & & & & & & b_{22} & c_{22}
\end{array}$$

It will be possible now to prove this theorem.

THEOREM. *A doubly infinite net of proper conics is the system of conic polars of a cubic, and this cubic is neither indeterminate nor impossible unless the net contains two conics consisting of doubly counting straight lines.*

PROOF. It has been shown by the particular matrix (7) that the cubic is determinate except under special conditions. These indeterminate cases may be considered in two classes:

(1) where not all conics of the net have contact with one another;

(2) where all the conics are tangent to one another, and of necessity in one common point.

Case (1). The equations and matrix above may be used. Erasing in succession the first 3 columns, we can factor the determinants by inspection. It appears that at least two of the quantities a_{23}, a_{13}, a_{12} must be zero in order that these determinants may vanish, and that this is sufficient also to make all the other 8-rowed determinants of the matrix vanish. Take $a_{13} = a_{23} = 0$. A linear combination of the three conic equations:

$$0 = (ax)^2 = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + a_{33}x_3^2$$
$$0 = (bx)^2 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2$$
$$0 = (cx)^2 = c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2$$

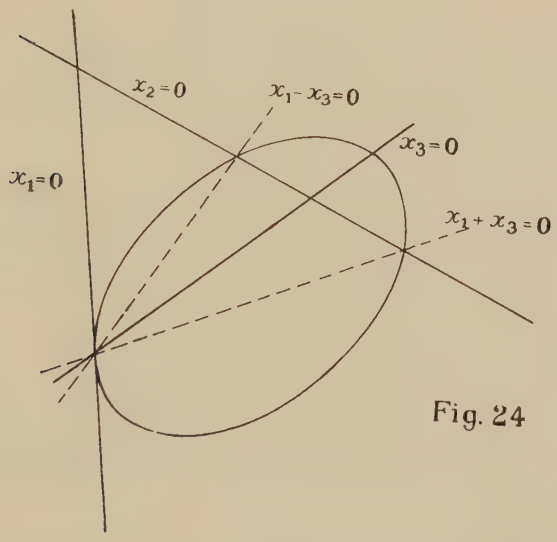


Fig. 24

can be determined so as to contain no term in x_3^2, and so as to be itself a perfect square, and this can be done in two ways. Hence in this case there must exist *two* doubly-counting lines.

Case (2). All conics of the net being in contact at one point, each degenerate conic of the net consists of the common tangent at that point together with some other line, unless

it be a pair of lines intersecting at the point of contact. Let one such have the equation:

$$(cx)^2 = x_1x_2 = 0$$

where $x_1 = 0$ is the common tangent. For the line $x_3 = 0$, take the polar of the point: $x_1 = x_2 = 0$ with respect to any proper conic of the net (Fig. 24), then evidently one other conic of the system is given by the equation:

$$(ax)^2 = x_1^2 - x_3^2 = 0 \text{ (dotted lines in figure).}$$

Considering now that the polar of the point of contact: $x_1 = x_3 = 0$, with respect to every proper conic of the net is the line: $x_1 = 0$, we see that a linearly independent third conic may be assumed to have the equation:

$$(bx)^2 \qquad x_1x_3 + b_{33}x_3^2 = 0.$$

From the forms $(ax)^2$ and $(bx)^2$ can be compounded linearly two perfect squares; that is, the involution of pairs of rays through the point of contact forms part of the net, and contains two doubly counting lines.*

It has been proven, then, that this singular net has all its conics in contact, and therefore contains two double lines; or otherwise, that the net is always the conic polar system of a determinate cubic except when it contains at least two double lines.

When the quantities $(\alpha), (\beta), (\gamma)$ of equations (1) and (2) are determined either completely or as functions of one or more arbitrary parameters, the form $\Phi(y, x)$ is known, and the equation of the cubic is found by replacing (y) by (x) in equation (1):

$$0 = \Phi(x, x) = (\alpha x)(ax)^2 + (\beta x)(bx)^2 + (\gamma x)(cx)^2$$

$$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right)f(x) = 3f(x).$$

* In this case the solution of equations (5) is generally impossible, but in a special case possible and indeterminate, when one double line coincides with the tangent, or when the two double lines coincide.

Convenient statements of the theorem of this section are the two following.* *A cubic is given by any three of its conic polars not contained in one sheaf. Any three conics with no more than three common points are polars of a cubic.* The examination of indeterminate cases has verified this further corollary: *If the net of polar conics of a given cubic contains two or more degenerate conics consisting of double lines, then and only then does the same net of conic polars belong to an indefinitely great number of other cubics.*

* This theorem was enunciated, without limitation and without proof by HERMITE in the Journal für r. u. a. Math., Vol. 57 (1860), pp. 373–4. Formal proof was furnished by GUNDELFINGER in the same Journal, Vol. 80 (1875), pp. 73–85.

Chapter VII.

Generation of the cubic by the methods of Chasles and Grassmann. Other methods.

§ 1. Chasles's derivation of a cubic from projective sheaves of lines and conics.

If there be a sheaf of conics and a sheaf of lines projectively related, the intersections of each line with its corresponding conic lie in a cubic. Let the conics and lines respectively be given by the equations:

$$(ax)^2 + \lambda(bx)^2 = f + \lambda g = 0,$$
$$(Ax) + \mu(Bx) = \varphi + \mu\psi = 0.$$

The sheaves are projectively related when the values of λ, μ are connected by a lineo-linear equation:

$$\lambda\mu + p\lambda + q\mu + r = 0.$$

From these 3 equations, required to be satisfied simultaneously, eliminate λ and μ. The result is:

$$\frac{(ax)^2}{(bx)^2} \cdot \frac{(Ax)}{(Bx)} - p\frac{(ax)^2}{(bx)^2} - q \cdot \frac{(Ax)}{(Bx)} + r = 0,$$

or this:

$$(ax)^2(Ax) - p(ax)^2(Bx) - q(bx)^2(Ax) + r(bx)^2(Bx) = 0.$$

Five points of this cubic are immediately seen to be the centre of the sheaf of lines: $\varphi = \psi = 0$, and the four fundamental points of the sheaf of conics: $f = g = 0$. For one conic of the sheaf passes through any one point in the plane, hence one passes through the centre of the sheaf of lines and meets there its corresponding line. And a line that passes through any one of the 4 points fundamental to the sheaf of conics must meet there its corresponding conic.

The relation of λ and μ is simplified by a different representation of the sheaf of lines. We have

$$\lambda = -\frac{q\mu + r}{\mu + p}$$

If now we set $\varphi = p\varphi' - r\psi'$, $\psi = \varphi' - q\psi'$, we find:

$$\varphi + \mu\psi = (\mu + p)\varphi' - (q\mu + r)\psi'.$$

The line: $\varphi + \mu\psi = 0$, is equally represented by:

$$\varphi' \quad \frac{q\mu + r}{\mu + p}\psi' \qquad \varphi' + \lambda\psi' \qquad 0.$$

The two sheaves are now given by the equations:

$$f + \lambda g \qquad 0, \qquad \varphi' + \lambda\psi' = 0.$$

The cubic has also a simpler form:

$$\begin{vmatrix} f & g \\ \varphi' & \psi' \end{vmatrix} \quad 0.$$

From this it appears that the same cubic is given by the modified equation:

$$\begin{vmatrix} f + l\varphi' & g + l\psi' \\ \varphi' & \psi' \end{vmatrix} = 0.$$

where l is an arbitrary linear form, $l = l_1x_1 + l_2x_2 + l_3x_3$. The cubic is accordingly also the locus of intersections of corresponding members in two new sheaves:

$$(f + l\varphi') + \lambda(g + l\psi') = 0, \text{ and}$$
$$\varphi' + \lambda\psi' = 0.$$

That is to say, the sheaf of lines remaining fixed, the sheaf of conics can be replaced by any one of a 3-fold infinity of other sheaves; for l_1, l_2, l_3, are arbitrary parameters.

THEOREM. *If a sheaf of conics and a sheaf of lines are related projectively, the points in which each conic intersects its corresponding line lie upon a curve of third order. The cubic remaining the same, for the sheaf of conics may be substituted any one of a* 3*-fold infinity of other sheaves of conics.*

This construction gives all possible proper cubics. Suppose a cubic given, whose equation shall be $F = 0$. To show that it can be generated by the method of Chasles, we choose arbitrarily either (1) four points of the cubic to be fundamental points of the sheaf of conics, or (2) one point for centre of

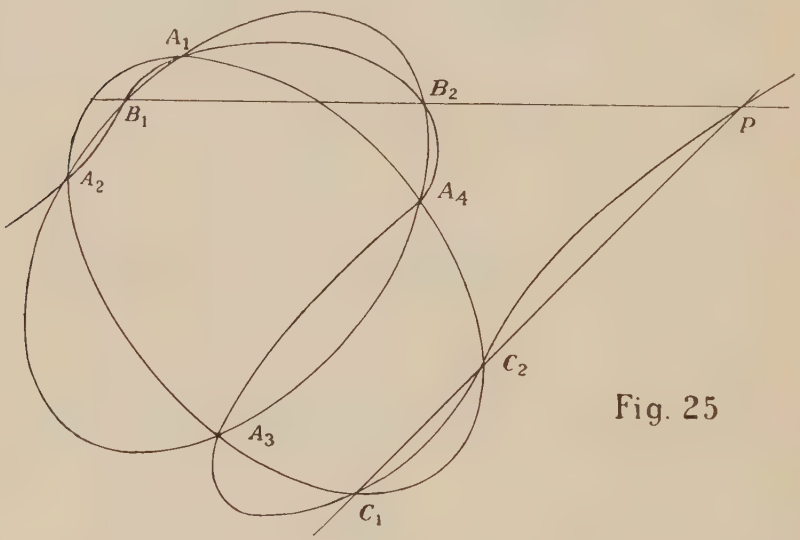


Fig. 25

the sheaf of lines and three fundamental points for the conics; then we need only to apply the theorem (II, § 3) that if a cubic contain 8 intersections of two other cubics, it contains also the ninth.

(1). Upon the cubic: $F = 0$, select any 4 points, A_1, A_2, A_3, A_4 (Fig. 25). Denote by $f = 0$, $g = 0$ any two conics through these 4 points, then $f + \lambda g = 0$ is any other conic of the sheaf. Let the pairs of other points in which the cubic is met by conics f and g respectively be denoted by B_1, B_2 and C_1, C_2. Produce the line (B_1, B_2), $(\varphi = 0)$ to meet the

cubic again in P, then the line (C_1, C_2), $(\psi = 0)$ also must contain P. For beside the cubic: $F = 0$, there is a second which contains the 9 points $A_1 A_2 A_3 A_4 C_1 C_2 B_1 B_2 P$, viz. the degenerate cubic: $g\varphi = 0$. All except P lie by construction upon a third cubic: $f\psi = 0$, hence this third cubic contains also the ninth point P. But as the conic: $f = 0$ contains 6 of these points, P must lie on the line $\psi = 0$. Therefore every member of the sheaf of conics meets some member of the sheaf of lines through P in two points of the cubic. It is to be shown that the two sheaves are projectively related in such a way that *corresponding* members intersect on the cubic.

Let any third conic have the equation: $f + \lambda_1 g = 0$, and let the line of the sheaf P which meets it on the cubic be given by: $\varphi + \mu_1 \psi = 0$, where $\mu_1 = c\lambda_1$. The two sheaves will be projectively related if we make correspond, for every value of λ, the two members:

$$f + \lambda g = 0, \quad \varphi + c\lambda\psi = 0.$$

And these corresponding members intersect on a cubic which must coincide with the given cubic, since they have in common 11 points: 4 fundamental for the conics, 1 for the lines, and the 3 pairs given by the values $\lambda = 0$, $\lambda = \lambda_1$, $\lambda = \infty$ (II, § 3).

(2) Select arbitrarily upon the cubic the points P, A_1, A_2, and A_3. Let the line: $\varphi = 0$ through P cut the cubic in B_1 and B_2: these two points together with A_1, A_2, A_3 determine a conic: $f = 0$, and that conic cuts the cubic in a sixth point A_4. Any conic: $g = 0$ through A_1, A_2, A_3, A_4 (Fig. 25) meets the cubic in 2 points, as C_1, C_2, whose join line: $\psi = 0$, must pass through P. The rest of the argument is the same as in case (1). The following theorem is thus proven.

THEOREM. *Any given proper cubic can be generated by the method of Chasles; and of the* 5 *fundamental points,* 4 *for the sheaf of conics and* 1 *for the sheaf of lines, any* 4 *may be taken arbitrarily upon the cubic, provided no* 3 *fundamental points for conics be taken collinear.*

§ 2. Three projective nets of lines generate a cubic.

In Schroeter's construction for points of the cubic curve (VI, § 6) there are concerned 3 conics. To every point of the curve there is related a second point, that in which its three polars with respect to the three conics intersect. Or we may say: to every point of the plane we determine three polars, all these polars constitute three projective systems or nets of lines, and the points which are common to three corresponding lines constitute a cubic curve. Now instead of polarity with respect to a conic we may employ skew reciprocity. To every point (y) of a plane there shall correspond a line in each of three skew reciprocal systems, whose equations may be written:

$$\begin{aligned} y_1(ax) + y_2(a'x) + y_3(a''x) &= 0, \\ y_1(bx) + y_2(b'x) + y_3(b''x) &= 0, \\ y_1(cx) + y_2(c'x) + y_3(c''x) &= 0. \end{aligned}$$

These 3 lines will generally not have a common point. When they have a point in common, the 3 equations are simultaneously satisfied, and (y) may be eliminated. The locus of the common point (x) is thus found, a cubic having the equation:

$$(1) \qquad \begin{vmatrix} (ax) & (a'x) & (a''x) \\ (bx) & (b'x) & (b''x) \\ (cx) & (c'x) & (c''x) \end{vmatrix} = 0.$$

If we seek the locus of points (y) whose 3 corresponding lines have a common point, that too is a cubic:

$$(2) \qquad \begin{vmatrix} a_1y_1 + a_1'y_2 + a_1''y_3, & a_2y_1 + a_2'y_2 + a_2''y_3, & a_3y_1 + a_3'y_2 + a_3''y_3 \\ b_1y_1 + b_1y_2 + b_1''y_3 & \dots\dots\dots\dots & \dots\dots\dots\dots \\ c_1y_1 + c_1y_2 + c_1''y_3 & \dots\dots\dots\dots & \dots\dots\dots\dots \end{vmatrix} = 0.$$

In Schroeter's construction these two cubics, (1) and (2), were coincident, and a point (x) of the one with a point (y) of the other constituted a pair. As (x) moved, describing the

entire curve, (y) also moved describing the entire curve, and the two never met unless the curve had a singular point (III, § 5). Here however the two curves are generally not the same, instead of a double locus there appear two single loci in a one-to-one point correspondence.

The equations for the skew reciprocities and for the cubics can be abbreviated if we write for a_i, $A_1 a_i$; for a_i', $A_2 a_i$; for a_i'', $A_3 a_i$; and similarly for coefficients (b), (c). For the reciprocities the relations become:

$$(Ay)(ax) = 0, \ (By)(bx) = 0, \ (Cy)(cx) = 0,$$

and the cubic equations are respectively:

$$(1) \qquad (ABC)(ax)(bx)(cx) = 0,$$

$$(2) \qquad (abc)(Ay)(By)(by) = 0.$$

THEOREM. *Three skew reciprocal relations determine in general two cubics, each point on the one being the pole of three lines which intersect in a point of the other.*

This mode of generating a cubic can be looked at differently, namely as a relation between two sheaves of conics. All values (x) that satisfy the equation (1), otherwise written thus:

$$(3) \qquad \begin{vmatrix} A_1(ax) & A_2(ax) & A_3(ax) \\ B_1(bx) & B_2(bx) & B_3(bx) \\ C_1(cx) & C_2(cx) & C_3(cx) \end{vmatrix} = 0$$

must make the constituents of the first row proportional to a linear combination, with constant coefficients, of the other constituents in their respective columns. That is, at every point (x) of the curve will vanish, for some value of λ, the 3 determinants of the array:

$$(4) \begin{vmatrix} A_1(ax) & A_2(ax) & A_3(ax) \\ B_1(bx) + \lambda C_1(cx) & B_2(bx) + \lambda C_2(cx) & B_3(bx) + \lambda C_3(cx) \end{vmatrix}.$$

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Such points (x) will be necessarily intersections of conics given by the same value of λ in the two sheaves:

$$(5)\quad \begin{vmatrix} A_1(ax) & A_2(ax) \\ B_1(bx) & B_2(bx) \end{vmatrix} + \lambda \begin{vmatrix} A_1(ax) & A_2(ax) \\ C_1(cx) & C_2(cx) \end{vmatrix} = 0 \quad \text{and}$$

$$\begin{vmatrix} A_1(ax) & A_3(ax) \\ B_1(bx) & B_3(bx) \end{vmatrix} + \lambda \begin{vmatrix} A_1(ax) & A_3(ax) \\ C_1(cx) & C_3(cx) \end{vmatrix} = 0.$$

Equations (5) are satisfied also, however, by certain points that do not reduce to zero the third determinant of the matrix (4), namely by points satisfying two linear equations:

$$(6)\quad A_1(ax) = 0 \text{ and } B_1(bx) + \lambda C_1(cx) = 0 \qquad (\lambda \text{ variable}),$$

i. e. by all points of the line: $A_1(ax) = 0$.

Conversely all intersections of corresponding conics in the two sheaves (5) will satisfy the cubic equation (3), except the points of the line: $A_1(ax) = 0$. Of the four fundamental points of each sheaf (5), one lies on this line, the others not. The locus of intersections of corresponding conics in the two sheaves would be a quartic but for this special feature, viz. conics that correspond have one intersection always upon the line joining a certain fundamental point of the one sheaf to a certain fundamental point of the other. In general the following theorem is readily proved.

THEOREM. *If a line be drawn joining a fundamental point of one sheaf of conics to a fundamental point of a second sheaf, but not containing any other fundamental point, then the two sheaves can be related projectively by taking as corresponding conics every pair that intersect each other on that joining line, and the locus of other intersections of such pairs will be a cubic passing through three fundamental points of each sheaf.* The converse theorem is also true, that all proper cubics can be generated in this way, as will be proven when the fundamental theorem on residues has been established. (Chapter VIII).

§ 3. Contacts between lines and conics of two sheaves.

Given a sheaf of lines: $(px)+\mu(qx)=0$, and a sheaf of conics: $(ax)^2+\lambda(bx)^2=0$. Each line is tangent to 2 conics, and each conic to 2 lines. *The locus of points of contact is a cubic.* Every contact must be considered as two intersections, and the locus of intersections will first appear as a sextic; whence if each contact be counted as a single point, it will be a cubic.

Fig. 26

Lines (μ) that touch a conic (λ) satisfy the condition:

$$\begin{vmatrix} a_{11}+\lambda b_{11} & a_{12}+\lambda b_{12} & a_{13}+\lambda b_{13} & p_1+\mu q_1 \\ a_{21}+\lambda b_{21} & a_{22}+\lambda b_{22} & a_{23}+\lambda b_{23} & p_2+\mu q_2 \\ a_{31}+\lambda b_{31} & a_{32}+\lambda b_{32} & a_{33}+\lambda b_{33} & p_3+\mu q_3 \\ p_1+\mu q_1 & p_2+\mu q_2 & p_3+\mu q_3 & 0 \end{vmatrix} = 0.$$

In this if we insert the values of λ, μ in terms of (x), the coordinates of the contact, the equation becomes:

$$\begin{vmatrix} a_{11}(bx)^2 - b_{11}(ax)^2, & a_{12}(bx)^2 - b_{12}(ax)^2, & & \\ & & a_{13}(bx)^2 - b_{13}(ax)^2, & p_1(qx) - q_1(px) \\ a_{12}(bx)^2 - b_{12}(ax)^2, & a_{22}(bx)^2 - b_{22}(ax)^2, & & \\ & \cdots\cdots\cdots & \cdots\cdots & \\ a_{13}(bx)^2 - b_{13}(ax)^2 & \cdots\cdots\cdots & & \\ & \cdots\cdots\cdots & \cdots\cdots & \\ p_1(qx) - q_1(px) & \cdots\cdots\cdots & & \\ & p_3(qx) - q_3(px), & 0 & \end{vmatrix} = 0.$$

To reduce this, multiply for example the third column by x_3, and add the first multiplied by x_1 and the second by x_2; then operate in the same way on the rows. Three constituents vanish identically so that four zeros fill a square in the lower right hand corner, and the equation takes the form:

$$\left\{ \frac{1}{x_3} \begin{vmatrix} a_1(ax)(bx)^2 - b_1(bx)(ax)^2 & a_2(ax)(bx)^2 - b_2(bx)(ax)^2 \\ p_1(qx) - q_1(px) & p_2(qx) - q_2(px) \end{vmatrix} \right\}^2 = 0.$$

This confirms the assertion that the locus is a doubly counting cubic.

Each line of the sheaf meets this locus at its two contacts with conics, and again in the centre of the sheaf of lines. Each conic meets the locus at its two contacts with lines, and again in the four fundamental points for conics; for it is easily seen that the locus passes through the fundamental points.

This mode of generation is equivalent to that of Chasles (VII, § 1). The contacts of lines from the centre (y) to a conic: $(ax)^2 + \lambda(bx)^2 = 0$, lie on the polar of (y), whose equation is: $(ay)(az) + \lambda(by)(bz) = 0$. All these polars constitute a second sheaf of lines with centre (y') at the point where $(ay)(ay') = (by)(by') = 0$, and this sheaf is projective to the sheaf of conics,—exactly the necessary data for Chasles's method (Fig. 26). And from the latter can be determined the former sheaf, for the two centres are conjugate points

with respect to all conics of the sheaf. Hence we conclude, as in (VII, § 1), the following theorem.

THEOREM. *Every proper cubic may be described as the locus of contacts of the conics of a sheaf with the lines of a sheaf. The fundamental points of both sheaves are on the cubic; and of these five fundamental points, four may be taken arbitrarily upon a given cubic.*

§ 4. Sheaves of lines in a (2 : 2) correspondence.

In the construction discussed in § 3 two sheaves of lines were considered as meeting the conics of a sheaf; the one a sheaf of polars, the other a sheaf of tangents. We may omit the conics and determine the cubics by intersections of each line of the one sheaf with two corresponding lines in the other. Each polar has as corresponding lines the tangents to its conic; and each tangent, since it touches two conics, has as correspondents the two polars taken with respect to those conics. One singularity must be noticed: the conic through the centre of lines (Fig. 26), will have the same line (y, y') for tangent and for polar of (y) simultaneously, hence this one line must be self-correspondent in the two sheaves.

To construct such a correspondence *ab initio*, take two sheaves of lines with different centres, in a two-and-two correspondence, with one self-corresponding ray (in the so-called *half-perspective* position relatively to each other). Denote this singular ray by: $(ax) = 0$, and the two sheaves respectively by

$$(1) \qquad (ax) + \lambda(bx) = 0 \text{ and } (ax) + \mu(cx) = 0.$$

Determine a (2 : 2) correspondence by a quadri-quadric equation between λ and μ, such that to $\lambda = 0$ will correspond $\mu = 0$:

$$(2) \qquad F(\lambda, \mu) = F_{22}\lambda^2\mu^2 + 2F_{21}\lambda^2\mu + 2F_{12}\lambda\mu^2 + 4F_{11}\lambda\mu + \cdots + F_{10}\lambda + F_{01}\mu = 0,$$

(the coefficient F_{00} being necessarily taken as zero). Between (1) and (2) eliminate λ and μ. The result is evidently of degree 4 in (x), but contains a factor (ax); that factor being omitted, the other factor gives the cubic locus. It has been shown that every cubic can be generated in this way, since

this is exactly equivalent to the method of § 3, and indeed it is plain from the fact that we have here 9 constants at our disposal, 2 in the choice of centres (y) and (y') upon a given cubic, and 7 in the coefficients F_{ik} in (2) after setting $F_{00} = 0$.

THEOREM. *If two sheaves of lines are in a* (2 : 2) *correspondence such that the ray joining their centres is self-correspondent, then the points where corresponding rays intersect* (*exclusive of points on the self-corresponding line*), *will lie on a cubic; conversely every cubic can be generated in this manner.**

A special case of such a correspondence is that in which corresponding lines are tangents from the two fixed centres to the same conic in a system of conics touching four fixed lines, e. g. in a system of confocal conics.†

§ 5. Centres of involutions of rays through six points.

Two pairs of points being fixed, the locus of a variable fifth point from which the four are projected by an harmonic pencil is a conic. For a cubic a similar definition is given by the use of the term *involution*. Two pairs of rays in a sheaf (or two pairs of points on a line) have always a common harmonic pair. If now a third pair of the same sheaf (or range of points) is divided harmonically by the same common pair, it is said to be in involution with the first two. In short, three pairs of rays in a sheaf form a pencil in involution if any other pair divides them all harmonically. To apply this notion, let three pairs of points: A, A', B, B', C, C', be fixed arbitrarily in a plane. From every point X of the plane project these points by three pairs of rays. The locus of points X in which the three pairs of projectors form pencils in involution is a cubic.

At first sight it would seem that 12 arbitrary constants are involved, 2 in the choice of each of the 6 points, whereas only 9 constants are required to determine a curve of the

* This mode of generating a cubic, suggested by STEINER, is used as the definition of a cubic by SCHROETER in his *Theorie der Kurven dritter Ordnung* (Leipzig, Teubner, 1880).

† See Schroeter's work cited above, p. 16.

third order. We must expect to find therefore that these 3 pairs are members of a simply infinite system of pairs of points; and that any center which projects 3 pairs by rays in involution will project all the other pairs by rays in the same involution. If this true, we shall have not 12 but 9 actual constants. The 6 points themselves evidently lie on the locus, since one ray through such a point is indeterminate; it would be the tangent to the locus at that point, and may be taken arbitrarily as the sixth ray of the pencil in involution determined by the rays joining it to the other fixed points.

To find directly the equation of the locus, we shall use most easily Clebsch's *method of bordering* a binary abbreviated formula. Instead of demonstrating the method in its generality, we shall give the complete details of this particular application, from which then a general statement and proof might be built up. First we consider an involution of 3 pairs of points on a line, determining the formal relation among their coordinates. Secondly we apply this result to the pencil of rays $X(AA', BB', CC')$ which join a variable point to 3 pairs of given points, and show thus that this locus of centres of involution is a cubic.

1. Let the coordinates (or parameters) of 6 points on a line be $(\alpha_1 : \alpha_2)$, $(\alpha_1' : \alpha_2')$, $(\beta_1 : \beta_2)$, etc., denoted by (α), (α'); (β), (β'); (γ), (γ'). If they are in involution, call the common harmonic points (t) and (t'). For harmonic division the conditions are:

$$(\alpha t)(\alpha' t') + (\alpha t')(\alpha' t) = 0 \quad \text{where } (\alpha t) \text{ denotes } (\alpha_1 t_2 - \alpha_2 t_1),$$
$$(\beta t)(\beta' t') + (\beta t')(\beta' t) = 0,$$
$$(\gamma t)(\gamma' t') + (\gamma t')(\gamma' t) = 0.$$

Expand these conditions, and eliminate the quantities $t_1 t_1'$, $t_1 t_2' + t_2 t_1'$, and $t_2 t_2'$.

$$(1) \qquad \begin{vmatrix} 2\alpha_1\alpha_1' & \alpha_1\alpha_2' + \alpha_1'\alpha_2 & 2\alpha_2\alpha_2' \\ 2\beta_1\beta_1' & \beta_1\beta_2' + \beta_1'\beta_2 & 2\beta_2\beta_2' \\ 2\gamma_1\gamma_1' & \gamma_1\gamma_2' + \gamma_1'\gamma_2 & 2\gamma_2\gamma_2' \end{vmatrix} = 0.$$

To simplify the reduction, consider that this is identically the

result of 3 polar operations upon a well-known determinant; namely it is expressed by:

$$\left(\alpha_1'\frac{\partial}{\partial\alpha_1}+\alpha_2'\frac{\partial}{\partial\alpha_2}\right)\left(\beta_1'\frac{\partial}{\partial\beta_1}+\beta_2'\frac{\partial}{\partial\beta_2}\right)\left(\gamma_1'\frac{\partial}{\partial\gamma_1}+\gamma_2'\frac{\partial}{\partial\gamma_2}\right)\cdot\begin{vmatrix}\alpha_1^2 & \alpha_1\alpha_2 & \alpha_2^2\\ \beta_1^2 & \beta_1\beta_2 & \beta_2^2\\ \gamma_1^2 & \gamma_1\gamma_2 & \gamma_2^2\end{vmatrix}=0.$$

Reducing the determinant and abbreviating the operators, we have this:

$$\left(\alpha'\frac{\partial}{\partial\alpha}\right)\left(\beta'\frac{\partial}{\partial\beta}\right)\left(\gamma'\frac{\partial}{\partial\gamma}\right)\cdot\{(\alpha\beta)(\alpha\gamma)(\beta\gamma)\}=0.$$

Carry out now the operations. The result is a sum of 8 terms, again susceptible of simpler form.

$$(2)\qquad \left\{\begin{array}{l}(\alpha\beta)(\alpha'\gamma)(\beta'\gamma')+(\alpha\beta)(\alpha'\gamma')(\beta'\gamma)\\ +(\alpha\beta')(\alpha'\gamma)(\beta\gamma')+(\alpha\beta')(\alpha'\gamma')(\beta\gamma)\\ +(\alpha'\beta)(\alpha\gamma)(\beta'\gamma')+(\alpha'\beta)(\alpha\gamma')(\beta'\gamma)\\ +(\alpha'\beta')(\alpha\gamma)(\beta\gamma')+(\alpha'\beta')(\alpha\gamma')(\beta\gamma)\end{array}\right\}=0.$$

This condition can be expressed in determinants each of which shall contain (e. g. like $(\alpha\beta')$) the coordinates of one first point and one second point from their respective pairs. Two terms are already such, and one factor in each of the other terms. Since now identically

$$(\alpha\beta)(\beta'\gamma')\equiv(\alpha\beta')(\beta\gamma')+(\alpha\gamma')(\beta'\beta)$$

the equation (2) becomes:

$$\left\{\begin{array}{l}(\alpha'\gamma)[(\alpha\beta')(\beta\gamma')+(\alpha\gamma')(\beta'\beta)]\\ \qquad\qquad +(\beta'\gamma)[(\alpha\alpha')(\beta\gamma')+(\alpha\gamma')(\alpha'\beta)]\\ +(\alpha\beta')(\alpha'\gamma)(\beta\gamma')+(\alpha\beta')[(\alpha'\beta)(\gamma'\gamma)+(\alpha'\gamma)(\beta\gamma')]\\ +(\alpha'\beta)[(\alpha\beta')(\gamma\gamma')+(\alpha\gamma')(\beta'\gamma)]+(\alpha'\beta)(\alpha\gamma')(\beta'\gamma)\\ +(\beta\gamma')[(\alpha'\alpha)(\beta'\gamma)+(\alpha'\gamma)(\alpha\beta')]\\ \qquad\qquad +(\alpha\gamma')[(\alpha'\beta)(\beta'\gamma)+(\alpha'\gamma)(\beta\beta')]\end{array}\right\}\qquad 0.$$

Cancel 6 terms, and remove the factor 4.

$$(3) \qquad (\alpha\gamma')(\gamma\beta')(\beta\alpha') + (\alpha'\gamma)(\gamma'\beta)(\beta'\alpha) = 0.$$

This equation (3) *is one of the simplest possible expressions of the involutory position of six points upon a line, or of six lines in a sheaf; the coordinates of the three pairs of points or lines being* (α), (α'); (β), (β'); *and* (γ), (γ').

2. Let two lines of reference through the variable point X or (x) have the coordinates (p) and (q), so that $(px) = (qx) = 0$, and by solution:

$$x_1 : x_2 : x_3 = (p_2q_3) : (p_3q_1) : (p_1 : q_2) \text{ or } (pq)_1 : (pq)_2 : (pq)_3.$$

Of the 6 given points in the plane, denote the coordinates by (a), (a'); (b) (b'); (c), (c'). The equation of the line (X, A) in running coordinates (y) is

$$(qa)(py) - (pa)(qy) = 0,$$

so that in the sheaf X we have for parameters of this line:

$$\alpha_1 : \alpha_2 = (pa) : (qa)$$

and for the other five similarly:

$$(pa') : (qa'); \ (pb) : (qb), \ (pb') : (qb'); \ (pc) : (qc), \ (pc') : (qc').$$

These values are to be substituted in equation (3), the condition for an involution. Consider the factor $(\alpha\gamma')$ of the first term.

$$(\alpha\gamma') = [(pa)(qc') - (pc')(qa)]$$
$$(pq)_1(ac')_1 + (pq)_2(ac')_2 + (pq)_3(ac')_3$$
$$x_1(ac')_1 + x_2(ac')_2 + x_3(ac')_3 = (xac).$$

The factor $(\alpha\gamma')$ *is replaced by the expression in ternary coordinates* $(ac'x)$. *This replacement is the bordering process introduced by Clebsch for passing from the binary domain to an invariant condition for a variable point or line* (x) *in the ternary domain.**

* See Cayley's paper in Liouville's Journal de Mathematiques, Vol. 9, p. 287; and Hesse's *Raum-Geometrie* edited by Gundelfinger, (1876), p. 106. Compare Schroeter's discussion in the Journal für r. u. a. Mathematik, Vol. 104, p. 71.

Completing the replacement, we have the desired equation:

$$(4) \qquad (xac')(xcb')(xba') + (xa'c)(xc'b)(xb'a) = 0.$$

Equation (4) *represents the locus of a point* (x) *from which three pairs of arbitrary points* (a), (a'); (b), (b'); (c), (c') *in the plane are projected by a pencil of rays in involution. That locus is therefore a curve of the third order.*

Schroeter's linear construction (VI, § 6) of points of a cubic from three pairs of corresponding points gives points upon the locus here determined. For if a point be taken at the intersection of lines (A, B') and (B, A'), then the pair of rays XA, XA' coincides with the pair XB', XB, and the pair XC, XC' is certainly divided harmonically by a pair of rays harmonic to XA, XA'. This locus contains therefore all points derived from the six by Schroeter's linear method, whence we have here again a direct proof that those points lie upon a cubic.

§ 6. Grassmann's method of generating a cubic, and its connection with Schroeter's.

Three fixed points and three fixed lines can be used for the determination of a cubic. To each point shall be assigned one particular line. Denote the points by A, B, C, and the lines assigned to them respectively by a', b', c'. From any other point X draw lines XA, XB, XC, and let XA meet a' in a point Q, also let XB, XC meet b' and c' respectively in R, S. Usually Q, R and S will not lie in a straight line, but for some positions of X they will be collinear. The locus of points X which make Q, R, and S lie in a straight line is a cubic.

THEOREM. *If a point moves in a plane in such a way that the lines joining it to* 3 *fixed points meet respectively* 3 *fixed lines in three collinear points, the point describes a cubic. This is Grassmann's method of describing a cubic.**

* H. GRASSMANN: *Ueber die Erzeugung der Curven dritter Ordnung* etc., Journal für r. u. a. Mathematik, Vol. 36 (1848), pp. 177-182. Also, *Grundzüge zu einer rein geometrischen Theorie der Curven* etc., ibidem Vol. 31 (1846), in particular p. 125.

Denote the coordinates as follows: the point by $(x_1x_2x_3)$, A by $(A_1A_2A_3)$, B by $(B_1B_2B_3)$, C by $(C_1C_2C_3)$, the line a' by $(a_1a_2a_3)$, b' by $(b_1b_2b_3)$ c' by $(c_1c_2c_3)$. Abbreviate the determinants as heretofore (§ 5), and denote the coordinates of the point where XA intersects a' (Fig. 27) in a similar way by $((xA)a)_1$, $((xA)a)_2$, $((xA)a)_3$; that is, let $(xA)_2a_3 - (xA)_3a_2 = ((xA)a)_1$, etc. The condition which makes Q, R, S collinear will be expressed thus:

$$(1) \qquad \begin{vmatrix} ((xA)a)_1 & ((xA)a)_2 & ((xA)a)_3 \\ ((xB)b)_1 & ((xB)b)_2 & ((xB)b)_3 \\ ((xC)c)_1 & ((xC)c)_2 & ((xC)c)_3 \end{vmatrix} = 0.$$

This equation is of degree 3 in the coordinates (x), the curve is therefore a cubic. Grassmann points out in the diagram (Fig. 27) a plain reason for this: the figure is constructed solely by the use of right lines and points, and exactly 3 variable lines of the figure are required to contain the point X.

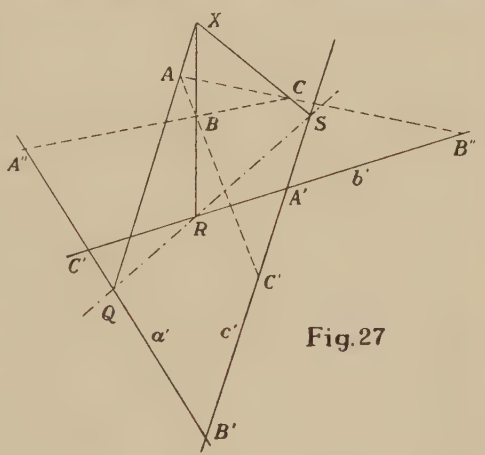

Fig. 27

Nine points of the locus are found by inspection. First the points A, B, C; for if X coincides with A, the direction XA is indeterminate, so that Q may be taken arbitrarily where SR (i. e. $B''C''$) meets a'. Secondly, the points A', B', C', in which the fixed lines a', b', c' intersect; for if X coincides with A', the points R and S must coincide with A', while Q will remain distinct, so that QA' contains both R and S. Finally, the points A'', B'', C'', lie on the locus, points in which corresponding sides of the fixed triangles intersect;

for if X coincides with A'', all three points Q, R, S come to lie on the right line BC. These 9 points are sufficient to determine the cubic completely, for a particular cubic containing 8 of them does not contain the ninth (II, § 3); viz. the lines $A''BC$, $A'B''C'$, and $A'B'C''$ constitute a cubic through 8 of the 9 points, but do not contain A, the ninth.

Compare this with Schroeter's linear construction, considering as pairs of corresponding points A, A'; B, B'; and C, C'. We see that the first operation would give points A'', B'' and C'', and therefore all later derivative points would lie on the cubic passing through these 9. But for further verification, notice that also the point B''' where AC' meets $A'C$ is found from both descriptions: from Schroeter's, as the point corresponding to B''; and by Grassmann's, because C' and A' lie already on the line b', upon which R is to be found by drawing $B'''B$. For in this way also a cubic would be determined passing through the same 12 points, $A\,B\,C$ $A'\,B'\,C'\,A''\,B''\,C''\,A'''\,B'''\,C'''$.

THEOREM. *The two fixed triangles employed in Grassmann's generation of the cubic have for vertices points which correspond in pairs in Schroeter's construction of the same cubic.**

Moreover since in Schroeter's construction two points of a pair are interchangeable, so in Grassmann's diagram it must be possible to interchange A with A', or B with B', without altering thereby the locus of the point X. Hence from the 3 pairs of points required by Schroeter we can arrange 4 pairs of triangles for Grassmann's method, and in each pair we may take the vertices of either one and the sides of the other. Accordingly *with the* 6 *fundamental points of a Schroeter's generation of the cubic are associated* 8 *ways of generating the same cubic by Grassmann's method.*

These latter statements are of elementary character. They can be verified by the reduction of equation (1) to a suitable form, as we shall see directly.

* This connection between the methods of Grassmann and Schroeter was explained by Clebsch, Math. Ann., Vol. V (1872), p. 424.

§ 7. Identity of Grassmann's cubic with the locus of centres of involutions.

In equation (1) (VII, § 6) the line coordinates (a), (b), (c) may be replaced by equivalents in terms of the coordinates of the vertices: (A'), (B'), and (C') in Fig. 27.

$$a_i = (B'C')_i,\ b_i = (C'A')_i,\ c_i = (A'B')_i,\ (i = 1, 2, 3).$$

Thereupon the determinant equation is seen to contain the factor $(A'B'C')$. Removing this, it is easy to bring the equation into the form:

$$(1) \quad (xBC')(xCA')(xAB') + (xB'C)(xC'A)(xA'B) = 0.$$

But this is identical with the equation (4) (VII, § 5), which gave the locus of the variable centre (x) of a involution of lines through 6 fixed points A, A'; B, B'; C, C'. Hence a theorem.

THEOREM. *If three fixed points A, B, C, and three corresponding fixed lines $B'C'$, $C'A'$, $A'B'$ are used by Grassmann's method for the generation of a cubic, then from every point X of that cubic the rays drawn to A and A', to B and B', to C and C' constitute respectively three pairs in involution.*

From the fact that the two equations, which we may call the involution equation and the Grassmann equation, are identical one can see that one way of generating a cubic by Grassmann's method gives rise to 3 other ways; for in an involution each pair of elements may be named in either order, or in 8 arrangements as first and second. Correspondingly of course equation (4) of § 5, Ch. VII is only one of four coordinate normal forms in which the condition of involution can be expressed.

From this same equation (1) may be inferred Schroeter's construction of points; for the equation is satisfied if X is on the two loci:

$$(xBC') = 0,\ (xB'C) = 0,$$

in other words, the intersection of lines BC' and $B'C$ is on the curve; and so for two other pairs of cross lines between

pairs of points. To see that the intersection of lines BC and $B'C'$ is on the curve we must reduce the equation to the form:

$$(1\,a) \quad (xBC)(xC'A')(xAB') + (xB'C')(xCA)(xA'B) = 0,$$

another of the 4 coordinate forms referred to above.

Schroeter's construction and Hesse's lemma call attention explicitly to the *complete quadrilateral*, and upon a property of that figure would be based a purely geometrical treatment of the cubic. It is well known that the lines of a complete quadrangle cut any secant in six points in involution. Dually, the three pairs of vertices of a complete quadrilateral are projected from any point by six lines in involution. Now in the diagram illustrating Grassmann's method (Fig. 28), the 3 sides of one triangle $A'B'C'$ together with the variable straight line QRS constitute a complete quadrilateral, hence the rays XA and XA', XB and XB', XC and XC' form a pencil in involution. This may be taken here as a geometrical explanation of the identity of the three principal modes of generating cubics treated in the foregoing sections.

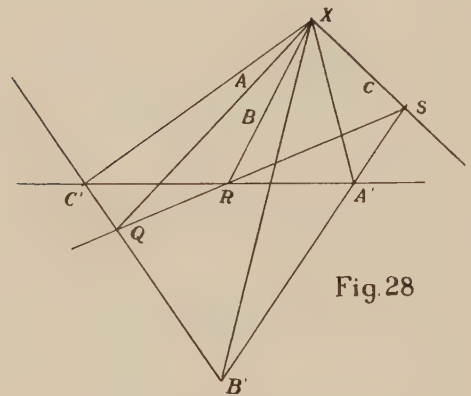

Fig. 28

Of the three constructions of Chasles, Schroeter and Grassmann, the last is simple, continuous, mechanical; the second has the advantage in simplicity, but does not give continuous series of points; while the first is inferior to both as regards simplicity, but is more readily applicable when a curve is given by data all of one kind, e. g. by nine of its points, as we shall see later.

Chapter VIII

Residual systems and corresidual systems of points on a cubic without double point.

§ 1. The number of arbitrary points upon a cubic through which a curve of order n can pass.

Curves of the third order have a property which has no analogue in the geometry of straight lines or of conics, but which is found in curves of all higher orders. Consider one curve as fixed, and a second curve, variable, intersecting the first; then the points of intersection may vary in position upon the first curve. If the fixed curve is a cubic, and the order of the variable curve is n, there are $3n$ points of intersection. All of these may vary simultaneously, or some part of them may remain fixed while the rest vary. What is the least number of intersections that can vary while the others are stationary? Is that least number the same for all values of n? On a straight line the intersections of a curve of the n-th order are given by the roots of an n-ic equation, and any one root may be varied while the $n-1$ others are left fixed. The same thing is true on a given conic. But if the fixed curve is a cubic, we have seen in Chapter II, § 3, that if $n = 3$, then 8 of the 9 intersections will determine the one remaining, the ninth. In other words, not less than two of the nine intersections can vary while the rest are stationary.

There are certain general propositions to be proved about *systems of points where a fundamental cubic curve, not degenerate, can be intersected by a curve of any order n.* Such propositions constitute the geometry of point sets on a cubic: or more briefly, *the geometry on a cubic curve.* For convenience, any $3n$ points in which the cubic is cut by a curve of order n (proper or degenerate) are called *a complete set* of points on the cubic, while any set not complete is called *partial.* Any

complete set of $3n$ points, when $n > 2$, is related not to a single curve of order n, but to an infinite system of such curves. For if $f(x) = 0$ denote the cubic, and $F(x) = 0$ any curve which cuts on the cubic a complete set of $3n$ points, then any other curve represented by the equation

$$\Phi(x) \equiv F(x) + M(x) \cdot f(x) \qquad 0$$

where $M(x)$ denotes any polynomial of order $n - 3$ in x_1, x_2, x_3, will cut out on the cubic the same complete set; for whatever point renders $f(x) = 0$ and $F(x) = 0$ will also make $\Phi(x) = 0$. As the factor $M(x)$ contains $(n-2)(n-1)/2$ or N terms with arbitrary coefficients, this is a system N-fold infinite.

The converse of this statement is also true, and is a theorem of prime importance. *If two curves of the same order, $F(x) = 0$ and $\Phi(x) = 0$, intersect the cubic, $f(x) = 0$, in the same complete set of points, then $F(x)$ can differ from some numerical multiple of $\Phi(x)$ only by a polynomial containing $f(x)$ as a factor.*

This can be proved independently, but is more conveniently included as a particular case in the much more general theorem:

If the points where a proper cubic is intersected by a curve, $f_n(x) = 0$, are all included in the set of points where the cubic is intersected by a second curve, $F_m(x) = 0$, of the same or of a higher order, then the equation of the second curve can be compounded linearly from those of the cubic and the first curve. That is, if the cubic equation is $\varphi(x) = 0$, then there is an identical relation

$$F_m(x) \equiv A_{m-n}(x) \cdot f_n(x) + B_{m-3}(x) \cdot \varphi(x).$$

The proof of this theorem requires a lemma which is the direct extension of that proved at the outset of § 1, Ch. II. As it is demonstrated by exactly similar argument, the statement will be sufficient.

LEMMA. *If a curve of order n intersects a straight line in $n+1$ points, the equations being respectively $f_m(x) = 0$ and $l_1(x) = 0$, then the line forms part of the curve, and there is an identical relation exhibiting l as a factor of f: $f(x) \equiv M(x) \cdot l(x)$.*

To make the proof of the main theorem complete, it would be necessary to admit all possible singularities in the cubic, all possible singularities or degeneracies in the curve f_n at common points, and all possible special relations of tangents at such points. We shall shorten the proof by considering only the simplest case, which furnishes a sufficient model for all the others taken in the order of their difficulty or intricacy. Assume therefore that

1. The curves, $\varphi = 0$ and $f = 0$, have all their intersections in distinct points, and that the triangle of reference is so situated that no vertex lies on any line joining two points of intersection. In this case the resultant of the equations $\varphi(x_1, x_2, x_3) = 0$ and $f(x_1, x_2, x_3) = 0$ found by eliminating x_3 will have $3n$ roots $(x_2 : x_1)$, all distinct.

2. The term x_3^3 occurs in the equation $\varphi(x_1, x_2, x_3) = 0$, with a coefficient not zero; that is, the curve of the third order does not contain the vertex $x_1 = x_2 = 0$ of the triangle of reference.

3. The elimination of x_3 between the equations $\varphi = 0$ and $f = 0$ has been performed, and the resultant expressed in the form

$$(1)\quad R(x_1, x_2) \equiv \alpha(x_1, x_2, x_3) \cdot f(x_1, x_2, x_3) + \beta(x_1, x_2, x_3) \cdot \varphi(x_1, x_2, x_3)$$

so that α and β contain no common factor rational in the coordinates.

Proof of the theorem. The product $\alpha \cdot F$ is to be formed, containing all three variables; then by ordinary division by φ, both dividend and divisor being arranged in descending powers of x_3, all powers of x_3 above the second can be removed (i. e. gathered into a product $\gamma(x_1, x_2, x_3) \cdot \varphi(x_1, x_2, x_3)$, so that the remainder is only of the second degree in x_3.

$$(2)\quad \alpha \cdot F = \gamma \cdot \varphi + G(x_1, x_2; x_3, x_3^2)$$

Now every point where $\alpha F = 0$ and $\varphi = 0$ will make $G = 0$. But this includes every point where $f = 0$ and $\varphi = 0$, by hypothesis, and also all those points where $\alpha = 0$ and $\varphi = 0$.

By the foregoing identity (1), these two sets of points include all those in which $\varphi = 0$ and $R = 0$.

The equation $R(x_1, x_2) = 0$ is a system of $3n$ lines converging to the point $x_1 = x_2 = 0$, each line giving one of the component factors of R. Any one such line cuts the cubic, $\varphi = 0$, in 3 points. Accordingly *there are* $3n$ *lines, each of which contains* 3 *points satisfying the equation, quadric in* x_3, $G = 0$. *Hence that curve,* $G = 0$ *by the lemma above stated, contains all the points of every such line*; and every factor of $R(x_1, x_2)$ must be a factor of $G(x_1, x_2; x_3)$. There subsists therefore the identity

$$G(x_1, x_2; x_3) \equiv R(x_1, x_2) \cdot A(x_1, x_2, x_3). \tag{3}$$

This with the identity (2) gives us

$$\alpha \cdot F \quad \gamma \cdot \varphi \; ; \; A \cdot R$$

or, by the use of identity (1)

$$\begin{aligned} \alpha \cdot F &\equiv \gamma \cdot \varphi + A \cdot (\alpha \cdot f + \beta \cdot \varphi), \\ F &\equiv A \cdot f + \frac{\gamma + A\beta}{\alpha} \cdot \varphi. \end{aligned} \tag{4}$$

The proof will be complete as soon as it is found that the quotient $\frac{\gamma + A\beta}{\alpha}$ is actually an integral expression, a polynomial in the coordinates x_1, x_2, x_3. Since F and $A \cdot f$ are integral expressions, in the identity (4) the term $(\gamma + A\beta)\varphi/\alpha$ is either integral or zero. By hypothesis the curve $\varphi = 0$ is a proper cubic, that is, the polynomial φ has no rational factors; hence α cannot divide φ, therefore α must divide exactly $\gamma + A\beta$. Call the quotient $B(x_1, x_2, x_3)$ or $B(x)$. The identity (4) is now written

$$F(x) \equiv A(x) \cdot f(x) + B(x) \cdot \varphi(x). \qquad \text{q. e. d.}$$

If the hypothesis be extended to admit contact of the curves $f = 0$ and $\varphi = 0$, we should have double factors in $R(x_1, x_2)$ or double lines through the vertex $x_1 = x_2 = 0$. In equation (2)

we should have now a stronger hypothesis both concerning intersections of $F = 0$ with $\varphi = 0$ and concerning those of $\alpha = 0$ with $\varphi = 0$. This could be treated either directly as a limit case, or indirectly through partial derivatives, and $G(x)$ would be found to have a double factor also. For full details of this and more complicated cases the reader should consult Noether's article in *Mathematische Annalen*, Vol. 6, pp. 351–9.

§ 2. Residual and corresidual sets of points.

The $3n$ points where a cubic curve is intersected by a curve of the n-th order are called a *complete point set* or a *complete group* on the cubic. Any set of points on the cubic, not complete, is called a *partial set*, or *group*. It may be partial because of the *number* being not a multiple of 3, or on account of the *positions* of the points although their number in $3n$, some multiple of 3.

The sum of two complete groups is a complete group. Let $f_1 = 0$ and $f_2 = 0$ denote the curves whose intersections with the cubic, $\varphi = 0$, are respectively the complete groups G_1 and G_2. Then the equation $f_1 f_2 = 0$ gives a curve intersecting in the sum of the two groups, $G_1 + G_2$. As this is a complete intersection of the cubic with another curve, it is a *complete group*. Note that there are usually an infinite system of curves having the same intersection, as shown by the equation $f_1 f_2 + M \cdot \varphi = 0$, where M is an arbitrary polynomial of the proper order.

When all the points of a group are points of a larger group, their removal is called subtraction, and the remaining points of the larger constitute the difference of the two groups.

If one complete group contains another, their difference also is a complete group. Suppose that two curves, $f_1 = 0$ and $f_2 = 0$, cut the cubic in the complete groups G_1 and G_2, and that G_1 contains G_2, the difference being a group G_3. Then is G_3 necessarily a complete group. For the theorem of the preceding section is applicable, and we have the identity:

$$f_1 = A \cdot f_2 + B \cdot \varphi.$$

Therefore every point where both $\varphi = 0$ and $f_1 = 0$ is either on the curve $f_2 = 0$ or on the curve $A = 0$; and conversely every point where $\varphi = 0$ and $A = 0$, that is, every point of G_3, is in the group G_1. Hence G_3 is complete, as asserted.

From a complete group a partial group may be removed or subtracted. Then the remainder cannot be complete (by the theorem just now proved).

Two partial groups whose sum is a complete group are called *residual* to each other. Denote two such partial groups and their sum by g_1, g_2, G.

$$G = g_1 + g_2 .$$

If a second curve contains all the points of g_1, and a different residual group g_3,

$$G' = g_1 + g_3 ,$$

then g_2 and g_3 are called *corresidual. Two groups that are residual to the same group are called corresidual.* Such groups may contain the same number of points, or different; and their points may be all different, or in part the same. The most important theorem concerning residues is this:

If two groups, g_1 and g_2, are both residual to a third, g_3:

$$G_1 = g_1 + g_3 , \qquad G_2 = g_2 + g_3 ,$$

and if g_1 is residual to a fourth, g_4:

$$g_1 + g_4 = G_3 ;$$

then is also g_2 residual to g_4:

$$g_2 + g_4 = G_4$$

where G_4 denotes some complete group. Or, if two groups are corresidual, every group residual to the one is residual to the other.

Proof. On the cubic curve, $\varphi = 0$, let the following com-

plete groups be cut by the curves whose equations are set opposite.

$$\begin{aligned} g_1 + g_3 \qquad & f_1 = 0 \\ g_2 + g_3 \qquad & f_2 = 0 \\ g_1 + g_4 \qquad & F_1 = 0. \end{aligned}$$

The sum of the second and third of these is a complete group, since it is the sum of two complete groups; and it includes the entire first group, $g_1 + g_3$, also complete; hence the remainder group, $g_2 + g_4$, is complete also; that is, g_2 is residual to g_4, the theorem to be proved.

§ 3. Two applications of the theory of residual sets.

One corollary of this theorem has been proved and used already. It is the fact that three cubics having eight points in common have in common also a ninth point. On this corollary rested further the proof that every proper cubic can be generated by the method of Chasles,—the intersection of corresponding members in a pencil of conics and in a projective pencil of lines.

The first of these we can state thus: a complete group of 9 points on a cubic is determined by any 8 of those points. Or otherwise: only one point on a cubic is residual to a group of 8 given points. A direct proof is as follows. If the given points be denoted by g_8, any second cubic through g_8 will cut the first in one additional point P. Let some straight line through P cut the first cubic also in $Q + R$, two additional points. Then $Q + R$ is residual to P, and corresidual to g_8. Hence any other point, P', residual to g_8 (that is, cut by any possible third cubic through g_8) must be residual also to $Q + R$. Hence $Q + R$ and P' form a complete group, three points on a line. But that line is determined by two points, Q and R, therefore P' must coincide with P, the third intersection of the cubic with the line QR by construction. How to construct this residual point P by the use of a rule or straight-edge alone is an important problem, one solution of which will be given here.

The other application of this theorem on corresidual groups,

the construction of the cubic by the method of Chasles, can be described thus. Four points on the cubic, a g_4, are chosen at random and made the basis, or common points, of a pencil of conics. Different conics of that pencil meet the cubic again in sets of two points, g_2, g_2', g_2'', etc. These latter sets are all residual to the g_4, and therefore corresidual to one another. But each of them determines a straight line, and thereby determines a residual point. Any one such point P being residual to g_2 must be residual also to the corresidual sets g_2', g_2'', etc. Hence the sets g_2 which lie on conics in the one pencil lie also upon corresponding lines in another pencil, the pencil of lines through P. The two pencils are projective to each other, as was proven in § 1, Chapter VII.

In the construction of a cubic curve by the method of Chasles, any four points of the curve may be chosen as basis points for a pencil of conics. The one point of the curve that is corresidual to those four must be taken as center of a projective pencil of lines. Three conics and their corresponding lines will fix the projectivity, then it remains to find, on every other line of the pencil, the two points where it is met by the corresponding conic. The locus of those pairs of intersections is the cubic.

§ 4. The ninth point determined by eight given points. Linear construction.

Two cubics intersect in nine points, and when any eight of these are given, the ninth is residual to those eight upon every cubic that contains them. Hence the ninth point can be found by choosing any one point, additional to the eight, to fix a particular cubic, and then finding on that particular cubic the point that is residual to the eight.

To simplify the problem, divide it. Consider the eight points as two groups of four. Determine one point corresidual to the first four and a second point corresidual to the second four, on some one particular cubic through the eight. These two points are then corresidual to the eight points; and it is easy to find the one point residual to those two, which must be residual also to the eight. That is, it will be the required ninth point.

Denote the given set of eight points by A_1, A_2, A_3, A_4; B_1, B_2, B_3, B_4, or $g_A + g_B$. Employ the Pascal theorem on conics upon each of the four conics determined by the g_A and B_1, g_A and $B_2, \ldots,$ g_A and B_4, in turn, to find their respective tangents in A_1. We require a point P corresidual to g_A. The lines PB_1, PB_2, etc. must form a pencil projective to the pencil of four tangents in A_1. Hence the locus of P is a conic through the four points g_B. This can be constructed by the Pascal method, after we first construct by the same method its tangent in one of those points, as B_1. That is, we must draw B_1B_2, B_1B_3, and B_1B_4, consider them related to the last three conics first constructed, and find the tangent in B_1 which makes the pencil projective to the four in A_1. So far only the eight given points have been used.

Next, replace any one, as B_4, by some selected point C, and proceed to find in exactly the same way another conic locus for P. That locus has in common with the former the three points B_1, B_2, B_3, hence the fourth can be found by linear construction. *That fourth common point of the conic loci is P, corresidual to the four points of g_A on the particular cubic that contains C.*

Repeat all these constructions, to find on the same particular cubic the point Q corresidual to the set g_B. Finally draw the line PQ, and the third point where it meets the particular cubic is the point desired. It is determined by considering PQ as a line in the pencil P, and seeking its intersection with the conic that contains g_A and the known point Q; or that of QP with the conic containing g_B and P.

Many steps of this construction can be abridged; e. g. we might take for P any arbitrary point on its locus, and make C coincide with it. Or we may take a new particular cubic, find a second couple of points P' and Q', then make lines PQ and $P'Q'$ intersect and give the required ninth point.

Special cases, in which three of the points are collinear, or where this happens more than once, give briefer solutions.

If the complete curve were supposed given by a drawing, so much of it that none of the points to be constructed should fall beyond its limits; and if the eight points upon the curve

were given, the finding of the required ninth point would be far simpler. Any two given points fix a line, and this produced sufficiently gives a third point on the cubic residual to the first two. In this way, from the eight points, four points are found which constitute a residual set or quadruple. Those four can be arranged in two pairs, and determine two points residual to the four, therefore corresidual to the given eight. From those two, finally, the one residual is determined, and this is residual also to the eight; it is the desired ninth point.

On residual and corresidual sets a general theorem can be formulated, whose proof is readily made from NOETHER's fundamental theorem.

On a cubic curve, if there are given a set of points not complete, in number $3n$, $3n+1$, *or* $3n+2$, *these are residual to other sets of* 3 *points,* 2 *points, or a single point. The last case is uniquely determinate; in the second, one of the two points may be chosen arbitrarily and the other is then uniquely determinate; in the first case, two are arbitrary and the third point then unique.*

PROBLEMS. Four lines form a quadrilateral $ABCD$. A cubic is tangent to AB at A, to BC at B, to CD at C, and to DA at D. Thus are given eight points on the cubic. Prove that the ninth, residual to these eight, is the intersection of the diagonals AC and BD.

Is this true also when the alternate sides of the above quadrilateral are arcs of two intersecting conic sections?

Let four of the eight given points be the common points of two conics, and the other four lie, two on each of those conics. What two lines meet in the ninth point?

CHAPTER IX.

Elliptic Integrals and Abel's Theorem.

§ 1. Parametric representation of points on a plane cubic by the use of an Elliptic Integral.

A differential expression $Mdx + Ndy$, when M and N denote functions of x and y, has a definite meaning and is integrable if x and y are coordinates of a point restricted to a definite curve as path of integration. An elliptic integral arises, for example, when N is zero and M is a rational algebraic function of x and y, and the path of integration is a plane cubic curve. Take the equation of the curve in the Weierstrassian normal form (see Chapter II, § 9),

$$y^2 = 4x^3 - g_2x - g_3.$$

The particular integral that we shall employ is

$$u = \int^{x,y} \frac{dx}{y} = \int^{x} \frac{dx}{\sqrt{(4x^3 - g_2x - g_3)}}.$$

The lower limit of the integral we shall fix arbitrarily. When it is taken to several different points of the curve as upper limits, as (x_1, y_1), (x_2, y_2), etc., the integral may be denoted by u_1, u_2, etc. with corresponding subscript indices.

$$u_1 = \int^{x_1,y_1} \frac{dx}{y}, \quad u_2 = \int^{x_2,y_2} \frac{dx}{y}, \quad \ldots\ldots$$

This integral is useful in geometrical questions by reason of its relation to corresidual point sets.

§ 2. Abel's Theorem.

The following is the famous Theorem of Abel as related to points on a plane cubic. The term *modulus* will be defined later.

ABEL'S THEOREM. *When the points which form the upper limits and those which form the lower limits of a set of elliptic integrals,*

$$u_i = \int^{x_i, y_i} \frac{dx}{y} \qquad (i = 1, 2, 3, \ldots, n)$$

are two corresidual sets of points on the plane cubic:

$$y^2 = 4x^3 - g_2 x - g_3,$$

then the sum of those integrals is zero, or differs from zero by the sum of integral multiples of two constants, ω_1 *and* ω_2, *called the moduli or periods of that integral* u:

$$u_1 + u_2 + \cdots + u_n = 0 + m_1\omega_1 + m_2\omega_2.$$

The converse also is true: if the sum of n such integrals is zero or a linear integral combination of the two moduli, then the n upper limit points form a set corresidual to the set of n lower limit points.

We shall fix as lower limits three collinear points, and for convenience they may be taken coincident, namely at a point of inflexion of the curve. Then we have to prove that: *If the three upper limit points are collinear, the sum of the three integrals is zero, or differs from zero by a sum of integral multiples of the two moduli.*

Lemma on partial fractions.

A formula often proved in treatises on Integral Calculus relates to the resolution of rational fraction in one variable into simpler fractions. Denote the fraction by $\frac{N(x)}{D(x)}$, and let the degree of D be n, that of N not higher than $n-1$. Also let the n linear factors of D be all different, and different from those of N:

$$D(x) = c(x - x_1)(x - x_2) \cdots (x - x_n).$$

The formula is then the following:

$$\frac{N(x)}{D(x)} = \sum_{i=1}^{n} \frac{N(x_i)}{(x - x_i)D'(x_i)},$$

Here $D'(x)$ denotes $\frac{\partial}{\partial x} D(x)$. (See Todhunter's *Integral Calculus*, § 19.)

For our present use, let the fraction be $\frac{x \cdot \varphi(x)}{F(x)}$, and $F(x) \equiv c(x-x_1)(x-x_2)(x-x_3)$. Then for the value $x = 0$ we shall find

$$\left[\frac{x \cdot \varphi(x)}{F(x)}\right]_{x=0} \equiv \left[\sum_{i=1}^{3} \frac{x_i \cdot \varphi(x_i)}{(x-x_i) \cdot F'(x_i)}\right]_{x=0}, \text{ or}$$

$$0 \equiv \frac{\varphi(x_1)}{F'(x_1)} + \frac{\varphi(x_2)}{F'(x_2)} + \frac{\varphi(x_3)}{F'(x_3)}.$$

This is the lemma here required. The proof here used is taken, with slight changes, from Weber, Elliptische Functionen, § 9.

Proof of Abel's Theorem for collinear points.

Let a straight line vary continuously with the variation of a parameter t. For $t = t_0$ let it intersect the cubic curve

$$y^2 = 4x^3 - g_2 x - g_3$$

in the points forming lower limits of the integrals

$$u_1 + u_2 + u_3,$$

and for $t = t_1$, in the upper limit points. Let the points remain collinear during integration, upon the line

$$ax + b + y = 0$$

a and b denoting functions of t as specified. The three varying values of x are found from the equations of line and cubic, after eliminating y, as functions of t:

$$F(x) = (ax+b)^2 - 4x^3 + g_2 x + g_3 = 0,$$

or abbreviated, $F = P^2 - f = 0$. At each intersection we have the differential relation (since $\frac{\partial f}{\partial t} = 0$)

$$2P\frac{\partial P}{\partial t}dt + \frac{\partial F}{\partial x}dx = 0.$$

$$\therefore du = \frac{dx}{\sqrt{f}} = -\frac{2\partial P/\partial t}{\partial F/\partial x}dt = \frac{2x\frac{\partial a}{dt} + 2\frac{\partial b}{\partial t}}{12x^2 - 2a^2x - g_2 - 2ab}dt$$

hence by the lemma, since $F(x) = 0$ gives the three values x_1, x_2, x_3, and since $\partial P/\partial t$ is of degree 1 in x, while F is of degree 3,

$$du_1 + du_2 + du_3 = \sum_{i=1}^{3}\left(\frac{\partial P/\partial t}{\partial F/\partial x}\right)_{x=x_i}\cdot dt = 0.$$

Integrating, we have

$$u_1 + u_2 + u_3 = \int_{t_0}^{t_1} o\cdot dt = 0. \quad \text{q. e. d.}$$

This proves the direct theorem, that the vanishing of the sum of three integrals is a necessary condition for the upper limit points to be coresidual to the lower limits. As to the converse, we shall assume *that x is a single-valued function of u.** Then since a third limit point collinear with the first two is one solution of the condition

$$u_1 + u_2 + u_3 = 0,$$

it is the only solution.

The proof for a larger number of points is exactly similar to this for three. We know already by the fundamental theorem of Noether that only one condition is to be satisfied in order that one set of n points may be coresidual to another set, and that condition is now found to be expressed as a linear equation in the parameters:

$$u_1 + u_2 + \cdots + u_n = 0.$$

* For the proof of this statement, consult BURKHARDT: *Elliptische Funktionen*, § 7.

§ 3. Periods of the elliptic integral u.

The curve defined by an algebraic equation contains only a simple infinity of real points, but a double infinity of imaginary points. Each variable coordinate has a double infinity of complex imaginary values:

$$x = x_1 - x_2\sqrt{-1}$$

both x_1 and x_2 denoting *real numbers*. To every such value of x the Weierstrassian equation of the cubic curve determines 2 values of y, complex or real or pure imaginary. The aggregate of pairs of values of x and y is therefore doubly infinite, and can be thought of *in terms of real points* only as a surface. Paths of integration for the elliptic integral are any continuous lines in such a surface. Two paths of integration between the same limit points are proved to give the same value of the integral *provided they can be deformed continuously, the one into the other, while remaining in the surface*; that is while consisting solely of points satisfying the equation of the curve.

In *Analysis situs* it is shown that two independent closed paths, or loops, exist on the surface (or curve), and that any two finite paths between fixed limits can differ from each other only by a finite number of loops of the two kinds.

If the curve is bipartite, consisting of one odd branch and one even branch, such independent paths can be defined simply enough. The first will be the complete odd branch; the other, any continuous path from an initial point on the odd branch to any point on the even branch, together with a return route consisting of the same values of x in reverse order, and the values of y conjugate to those first employed. *The integral u taken over the former (real) path we shall call ω_1, that over the second (imaginary) path will be designated by ω_2.*

To make the matter more explicit for this bipartite cubic, let a line vary with one point fixed, namely the point at infinity on the y-axis. That is a point of inflexion, and we know (III, § 6) that exactly three tangents can be

drawn from it to other points of the curve, also that the three points of contact will all lie on the x-axis, since that is the harmonic polar of $(0, 1, 0)$ with respect to the curve $y^2 = 4x^3 - g_2x - g_3$. But the points on the x-axis, call then $x = e_1$, $x = e_2$, $x = e_3$, are given by the equation

$$4x^3 - g_2x - g_3 = 0 = 4(x - e_1)(x - e_2)(x - e_3).$$

If $g_2 > 0$, and all the roots are real, then the following diagram shows schematically the situation, the roots arranged in order of magnitude with $e_1 < e_2 < e_3$.

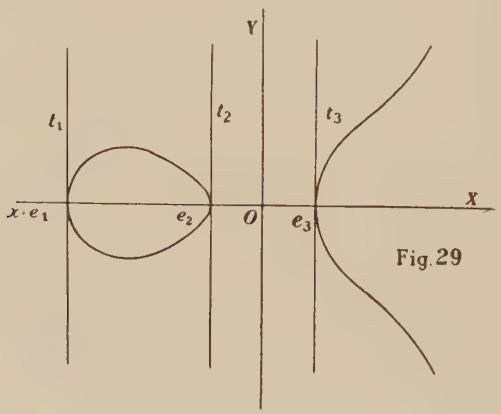

Fig. 29

The point of inflexion at ∞, $(x = 0, y = \infty)$ is the point selected for lower limit of the integral or parameter u

$$u = \int_{\infty}^{x, y} \frac{dx}{y} = \int_{0}^{x} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}.$$

Hence by Abel's Theorem every ordinate meets the curve in two finite points, the sum of whose parameters is zero or a period:

$$0 = u_2 + u_3, \text{ or } m_1\omega_1 + m_2\omega_2 = u_2 + u_3.$$

On any one of the three tangents, $u_2 = u_3$, because the points coincide.

This is obvious at e_3, since the paths of integration might well be from ∞ to $x = e_3$, along the upper half of the odd branch for u_2, along the lower half for u_3, and therefore with opposite values of y, so that $u_2 = -u_3$, meeting at $x = e_3$. If then $u_2 = \frac{\omega_1}{2}$, $u_3 = -\frac{\omega_1}{2}$. But as the line describes the *real* pencil about the center at $y = \infty$, while it passes from t_3 to t_2 the limit points which it carries are conjugate pure imaginary, and come to coincidence first at $x = e_2$. Call the increment of u_2 from e_3 to e_2, $\frac{\omega_2}{2} = B\sqrt{-1}$, then will the increment of u_3 be $-B\sqrt{-1}$, and the sum will have become

$$(u_2 + u_3)_{x=e_2} = \left(\frac{\omega_1}{2} + \frac{\omega_2}{2}\right) + \left(-\frac{\omega_1}{2} - \frac{\omega_2}{2}\right) = 0.$$

Again, fix one point of a variable line at $(e_2, 0)$ and let one intersection with the curve vary from $(0, \infty)$, along the positive half of the odd branch, to the point $(e_3, 0)$. The other intersection meantime will move along the negative half of the even branch from $(e_2, 0)$ to $(e_1, 0)$. The parameter of the former increases by $\frac{\omega_1}{2}$, hence that of the second changes from $\frac{\omega_1}{2} + \frac{\omega_2}{2}$ to $\frac{\omega_2}{2}$. The periods may therefore be indicated thus, for our present purposes.

$$2\int_{x=e_1}^{x=e_2} \frac{dx}{+y} = +\omega_1, \quad 2\int_{e_2}^{e_3} \frac{dx}{y} = \omega_2, \quad 2\int_{e_3}^{\infty} \frac{dx}{+y} = -\omega_1.$$

When the curve is unipartite, (having no even real branch), periods can be defined in this same way by reference to the intersections of the curve with the x-axis, two imaginary and one real. In this case, of course, there is not the possibility of defining the paths from e_1 to e_2 and from e_2 to e_3 by using exclusively real values of x. For geometrical applications, however, any path is as suitable as any other.

§ 4. Application of Abel's Theorem to brief demonstration of geometrical propositions.

The form of equation used above, the Weierstrassian normal equation of the cubic curve, represents not a particular curve, but equally well all curves of the third order. Hence the representation of its points by the values of an elliptic integral as parameter is equally valid for every such curve, *except those having singular points.* In case there is a double point or a cusp, the quantity $f(x)$ under the radical has a square factor, and the integral is no longer elliptic. We shall exclude singular cubics from consideration in this chapter.

We shall assume as lower limit point of the integral or parameter always a real point of inflexion. As there are three of these on a real cubic, this is permissible. A point of inflexion, thrice counted, forms a complete point set, namely the triple intersection of the curve with an inflexional tangent. Then of course if we are considering a sum of six, or of nine, integrals, the lower limits form a complete set, the intersection of the curve with that same inflexional tangent counted doubly, or triply, etc. *By this convention we have the vanishing of a sum of* $3n$ *values of the parameter as the necessary and sufficient condition for the* $3n$ *points* at *the upper limits to form a complete set, the complete intersection of the cubic with a curve of the nth order, proper or degenerate. Vanishing is to be understood as becoming equal to zero or to a sum of entire multiples of the two periods.*

Tangent at a given point.

At any point $u = u_1$, draw a tangent. Let it cut the curve again at $u = u_2$. Then the equation of condition is

$$2u_1 + u_2 = 0 + m_1\omega_1 + m_2\omega_2,$$
$$u_2 = -2u_1 + m_1\omega_1 + m_2\omega_2.$$

The tangent in the point u_1, *meets the curve also in the point* $-2u_1$.

Tangents from a given point of the curve.

If from a point u_2 any tangent is drawn, touching the curve at a point u_1, since

$$u_2 + 2u_1 = 0 + m_1\omega_1 + m_2\omega_2,$$
$$u_1 = -\frac{u_2}{2} + \frac{m_1\omega_1}{2} + \frac{m_2\omega_2}{2}.$$

Since entire periods added to the parameter do not change the point, the only possible points of contact are found by giving m_1 and m_2 the values 0 or 1. There are therefore four points:

$$-\frac{u_2}{2},\ -\frac{u_2}{2} + \frac{\omega_1}{2},\ -\frac{u_2}{2} + \frac{\omega_2}{2},\ -\frac{u_2}{2} + \frac{\omega_1 + \omega_2}{2}.$$

The four points of contact of a pencil of tangents drawn from a point of the cubic curve have parameters whose differences are half-periods.

Points of Inflexion.

For three intersections of a line with the cubic to coincide, the condition is

$$3u = 0 + m_1\omega_1 + m_2\omega_2,$$
$$u = m_1\frac{\omega_1}{3} + m_2\frac{\omega_2}{3}.$$

As the solutions are all given by assigning to m_1 and m_2 values 0, 1, or 2, there will be exactly nine inflexions. We may arrange the values of m_1 and m_2 in a square array:

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

The parameters will be found to sum up to entire multiples of ω_1 and ω_2 if we take three points in a horizontal line or in a vertical line of this array, or three whose positions correspond to any term in a three-by-three determinant. This is exactly similar, we notice, to the array used in Chapter IV. § 5, to represent trilinear coordinates of these same points.

If ω_1 denotes the *real* period, $u = 0$ being a real inflexion on the odd branch, we can verify here the statement that there are three, and no more, real inflexions. For, first, only the inflexion-points

$$0, \quad \frac{\omega_1}{3}, \quad \frac{2\omega_1}{3}$$

are on the odd branch. Secondly, if there are other real points of inflexion, their parameters differ from real values by $\frac{\omega_2}{2}$. But $\frac{\omega_2}{2}$ subtracted from any of those in the above array will leave some fraction of ω_1 plus an integral multiple of $\frac{\omega_2}{6}$ but not of $\frac{\omega_2}{2}$, which therefore is not a *real* quantity.

Inscribed Quadrilaterals.

If four lines intersect one another only in points of the cubic, they form an inscribed quadrilateral. Call the parameters of those intersections $u_1, u_2, \ldots, u_6$. The conditions of collinearity are

$$\begin{aligned} u_1 + u_2 + u_3 &\equiv 0 \\ u_1 + u_5 + u_6 &\equiv 0 \\ u_2 + u_4 + u_6 &\equiv 0 \\ u_3 + u_4 + u_5 &\equiv 0 \end{aligned}$$

the sign $\equiv$ signifying equality save for finite integral multiples of ω_1 and ω_2, called usually a *congruence*.

The points u_1 and u_4 are at opposite vertices of the quadrilateral, as are u_2 and u_5, u_3 and u_6. The difference of two of these equations of condition gives one of three congruences, or collectively:

$$u_1 - u_4 \equiv u_2 - u_5 \equiv u_3 - u_6 \equiv u_4 - u_1 \equiv u_5 - u_2 \equiv u_6 - u_3.$$

It follows that

$$u_1 - u_4 \equiv \frac{m_1\omega_1 + m_2\omega_2}{2},$$

so that *opposite vertices have parameters which differ by a*

linear combination of half periods, and all three pairs of opposites give congruent differences of parameters.

Hence for a given point u_1, the opposite parameter u_4 can be determined in three ways. For the solution $m_1 = 0$, $m_2 = 0$ would make the points coincide; this leaves only three pairs of values for (m_1, m_2), viz. (0,1), (1,0) and (1,1). Any other pair would give a parameter congruent to one of these.

Since there are no other inscriptible complete quadrilaterals than these three, and since any one of the three Hessian correspondences gives a system of inscribed complete quadrilaterals, it is evident that these systems are the same as those. *The pairs of points that form a Hessian correspondence on the cubic have parameters, whose constant difference is a half period; and the three incongruent half periods* $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, and $\frac{\omega_1 + \omega_2}{2}$ *belong respectively to the three Hessian correspondences.*

The convenience of this elliptic parameter for geometry on the cubic is strongly indicated if we compare the theorem on corresponding points in a Hessian relation, and its proof, with the following formulation.

If two pairs of points on the cubic are taken for pairs of opposite vertices in a complete quadrilateral, each pair having parameters which differ by the same half period, then the third pair of vertices will lie on the cubic, and their parameters will have the same difference. Denote the given parameters by

$$a,\ a + \frac{\omega}{2},\ \text{and}\ b,\ b + \frac{\omega}{2};$$

then, since

$$-(a + b) \equiv -(a + \frac{\omega}{2} + b + \frac{\omega}{2}),$$

two join-lines intersect the cubic in the same third point; and the other two join-lines do the same, giving the parameter:

$$-\left(a + \left(b + \frac{\omega}{2}\right)\right) \quad -\left(\left(a + \frac{\omega}{2}\right) + b\right).$$

This proves the theorem.

§ 5. Steiner's inscribed polygons.

If two points A and B on the cubic are taken as fundamental, a broken line inscribed in the cubic may be constructed thus: To any third point P_1 draw a line AP_1, and produce it to meet the cubic again in a point Q_1. Let BQ_1 meet the curve also in P_2, and AP_2 likewise in Q_2, etc. Alternate segments pass through A, the others through B. *Any two consecutive points in the series* $P_1, P_2, P_3 \ldots P_i, P_{i+1}, \ldots$ *will have parameters whose difference is* $b - a$, *the same as for fundamental points* B *and* A; *also for the series* $Q_1, Q_2, \ldots$ *the difference will be* $a - b$. *For* P_i *and* P_{i+n} *the difference is* $n(b-a)$.

The most interesting series are those which close, the point P_{n+1} coinciding with P_1. For that it is necessary and sufficient that

$$p_{n+1} - p_1 = n(b-a) = 0, \text{ or } b - a = \frac{m_1\omega_1 + m_2\omega_2}{n}.$$

Let A be given, with its parameter a; and suppose it is required that the polygon shall close with n pairs of points P_i, Q_i; also let n be a prime number. Then for each of the integers m_1, m_2, there are n possible choices that give incongruent solutions, $0, 1, 2, \ldots n-1$, save that we may not choose 0 for both. This gives therefore $(n^2 - 1)$ values for $b - a$; that is, $(n^2 - 1)$ positions of the other fundamental point B. When $n = 2$, the case of a quadrilateral, $n^2 - 1 = 3$ as we have seen above in the Hessian correspondences.

When the fundamental points are suitably fixed, any other point of the curve may be P_1. *The fixing of* A *and* B *as above described arranges all other points of the curve in cyclic sets of* n *points.*

It is an easy exercise to prove that if one of the points Q_i be chosen for P_1, the two sets (P) and (Q) are simply exchanged and each set has its order reversed. This also is seen directly from the given mode of construction.

When n is not a prime, certain other pairs beside $(0, 0)$ are to be rejected as values for (m_1, m_2). E. g. if n is 15, m_1, and m_2 must not both be multiples of 3, nor both multiples of 5, since that would determine a series closing in fewer than 15 repetitions.

The fundamental points A and B of a Steiner inscribed polygon are consecutive points P_i, P_{i+1} in a particular set of points P, alternate vertices of such a polygon; and therefore of course B and A are consecutive points in a Q-set. Every P-set is a Q-set in inverted order.

Conversely. *If in any closed Steiner polygon of n sides two alternate vertices P_1, P_2 be selected for A and B, fundamental points of a new Steinerian construction, the latter also will close with $2n$ sides.* For the only essential fact is the difference of parameters $p_2 - p_1$, which is in this case the same as the difference $b - a$. The same fact shows that:

The projection of a Steiner pair A, B, from any point C upon the cubic yields a new Steiner pair A', B' which determine Steiner polygons of the same number of sides. For if a, b, c, denote the parameters of A, B, and C, the projection A' of A from C will have the parameter

$$a' \equiv -(c + a)$$

and B' will have the parameter

$$b' \equiv -(c + b)$$

so that $b' - a' = (a - b)$, and that is by hypothesis

$$\frac{m_1 \omega_1 + m_2 \omega_2}{n},$$

the necessary condition for closure in $2n$ sides.

Any point of the P-set may be used as center C, and all the Q-set be projected into a set of n points (R). From all points of the P-set the projection of the Q-set will give the same set (R). The relation of the three sets of n points, (P), (Q), (R), is entirely symmetrical. Though every point of each set there can be drawn n lines, each containing one point of each of the other sets. The proof of these statements is immediate from the use of the elliptic parameters.*

* For fuller statements, and proofs by the notation of E. Weyr, see article by P. H. Schoute, in Journal für r. u. a. Mathematik, Vol. 95 (1883), p. 106.

§ 6. P. Serret's Pentagon construction for the cubic.

Can a pentagon be inscribed in a cubic so that every point where a side meets the opposite diagonal shall be a point on the curve? If so, denote the elliptic parameters of the vertices by $u_1, u_2, \ldots, u_5$; and the five *extra vertices*, or intersections of sides with diagonals, by $v_1, v_2, \ldots, v_5$. Number the latter thus: let u_1u_2 meet u_3u_5 in v_4, u_2u_3 meet u_4u_1 in v_5, etc. The ten equations of condition are as follows:

$$\begin{aligned}
u_1 + u_2 + v_4 &\equiv 0 & u_3 + u_5 + v_4 &\equiv 0\\
u_2 + u_3 + v_5 &\equiv 0 & u_4 + u_1 + v_5 &\equiv 0\\
u_3 + u_4 + v_1 &\equiv 0 & u_5 + u_2 + v_1 &\equiv 0\\
u_4 + u_5 + v_2 &\equiv 0 & u_1 + u_3 + v_2 &\equiv 0\\
u_5 + u_1 + v_3 &\equiv 0 & u_2 + u_4 + v_3 &\equiv 0.
\end{aligned}$$

These 10 conditions are not independent, for those in the left column and those in the right have the same sum: $2\sum_1^5 u_i + \sum_1^5 v_i \equiv 0$. Hence they do not define a particular set of 10 points on the curve, nor can they determine a finite number of such sets.

Subtract each equation on the right from the one opposite on the left. The results can be arranged thus:

$$u_2 - u_3 \equiv u_5 - u_1 \equiv u_3 - u_4 \equiv u_1 - u_2 \equiv u_4 - u_5 \equiv d.$$

Hence the parameters of the vertices form an arithmetical progression. Since the cycle is closed, i. e. since $u_1 + 5d \equiv u_1$, we require $5d \equiv 0$ or $d \equiv n_1\frac{\omega_1}{5} + n_2\frac{\omega_2}{5}$, a fifth part of a period. From this it follows that

$$u_2 \quad u_1 + d, \quad u_3 \quad u_1 + 2d, \quad u_1 \quad u_1 + 3d, \quad u_5 \quad u_1 + 4d.$$

The two equations for v_1 give the same solution

$$2u_1 + 5d + v_1 \equiv 0, \text{ or } v_1 \equiv -2u_1.$$

This shows that the tangent at u_1 meets the cubic again in v_1: the same can be shown for u_2 and v_2, u_3 and v_3, etc. Accordingly

$$v_2 - v_1 \equiv 2(u_1 - u_2) \equiv 2d.$$

But $2d$, like d, is one-fifth of a period of the elliptic parameter. We have proved this theorem:

Every point of the cubic may be taken as one vertex of a complete pentagon, whose ten vertices all lie on the curve. Five of the vertices, called extra, will be the tangential points of the other five called primary; and these extra vertices may be taken in their order as primary vertices for a new pentagon whose extra vertices will also lie on the same cubic; and this construction may be repeated indefinitely.

The theorem of P. Serret* comprises the above and its converse, or complement, as follows.

Any five points, no three being collinear, may be taken as primary vertices of a pentagon in any assigned order. The cubic curve which passes through those five primary vertices and four of the extra vertices will pass also through the fifth extra vertex of the complete pentagon.

For proof, write nine of the foregoing equations of condition, and in the remaining equation—one of those containing v_5—replace v_5 by v_6. Adding the right and left columns separately gives the proof that $v_6 \equiv v_5$, so that the final intersection lies on the curve.

§ 7. Triply tangent conics.

Abel's theorem states that for a conic, as well as for a line, the sum of the elliptic parameters of the points where it intersects a cubic is zero or a period (sum of integral multiples of ω_1 and ω_2), and the converse. If the six points coincide, two and two, so that the conic is tangent in three points to the cubic, say in u_1, u_2, and u_3, the condition to be satisfied is

$$2u_1 + 2u_2 + 2u_3 \equiv 0 = m_1\omega_1 + m_2\omega_2,$$

$$u_1 + u_2 + u_3 = m_1\frac{\omega_1}{2} + m_2\frac{\omega_2}{2}.$$

* P. Serret: Comptes Rendus, (Paris), Vol. 115 (1892), p. 406.

If m_1, m_2 have the values 0, 0 or are both even integers, the three points are on a line and the conic is that line twice counted. Of proper conics there are therefore 3 systems, corresponding to the three pairs of values for m_1, m_2: (0,1), (1,0), and (1,1).

In each of these 3 systems, two of the points may be taken arbitrarily, and the third is thereby completely determined.

The tangential points of the 3 contacts are collinear. Let v_1, v_2, v_3, denote their respective parameters. Since $v_1 + 2u_1 \equiv 0$, etc., it follows that $v_1 + v_2 + v_3 \equiv -(2u_1 + 2u_2 + 2u_3) \equiv 0$, and this is the Abelian condition for collinearity of the tangential points.

These 3 systems of conics have been described previously (Chapter III, § 14) in connection with poloconics.

Two points of contact may coincide. If this point of hyperosculation is known, u, and that of ordinary contact, v, is to be found,

$$2v \equiv -4u + m_1\omega_1 + m_2\omega_2;$$

there is only a half-period arbitrary, so that there are four solutions, the same as for linear tangents.

But if the ordinary contact is given, the other to be found,

$$u \equiv -\frac{v}{2} + m_1\frac{\omega_1}{4} + m_2\frac{\omega_2}{4},$$

then there are 16 apparent solutions of the problem. Of these, however, 4 belong to tangents twice counted, so that only 12 are proper solutions.

§ 8. Sextactic points.

Finally a conic may be sought which shall have all six points of intersection with the cubic coincident. Points where this contact of highest order can occur are called sextactic

points. Points of inflexion are excluded, since the conics belonging to them are only the inflexional tangents twice counted. In a sextactic point u the condition to be satisfied is

$$6u \equiv m_1\omega_1 + m_2\omega_2,$$

giving 36 apparent solutions. Deduct 9 for inflexions, and we find as the number of proper sextactic points 27. Note that 3 of these are on the odd branch of the cubic; and if there is an even branch, it will contain 6 such points. The former have as values for m_1, m_2 respectively: (1,0), (3,0), (5,0); the latter have: (0,3), (1,3), (2,3), (3,3), (4,3), and (5,3).

It is seen thus that these 9 *real sextactic points form a complete point set on the bipartite cubic.*

Chapter X.

Certain invariants, and covariant curves and systems.

§ 1. Covariant curves of the third order.

The fundamental cubics, $f(x_1, x_2, x_3)$ or f, and its Hessian H are two cubic covariants that we have studied already. There are no others of the third order which are not expressible as linear combinations of these two. Of such linear combinations, we can form as many as we please that shall be covariants, by using for multipliers (coefficients) pure numbers and invariants. Two invariants have been defined, S and T, of degrees 4 and 6 respectively in the coefficients of the fundamental cubic, while the degrees of f and H are respectively 1 and 3. If we wish a homogeneous covariant of the third order, write the expression

$$mS^\alpha T^\beta f + nS^\gamma T^\delta H$$

and the condition $4\alpha + 6\beta + 1 = 4\gamma + 6\delta + 3$. This is the same as

$$2(\alpha - \gamma) = 3(\delta - \beta) + 1.$$

If we remove from the covariant a factor $S^\gamma T^\beta$, or else $S^\alpha T^\delta$, (leaving only positive exponents), the essential covariant may be written

$$mS^{3k+2}f + nT^{2k+1}H,$$

or else

$$mT^{2k+1}f + nS^{3k+1}H,$$

where k may take all positive integral values. This amounts to saying that the ratio of the coefficients of f and H can be taken any numerical multiple of any power of S^3/T^2, into

the quantity $S^2 . T$ and the result will be, if freed of fractions, a rational integral cubic covariant of the fundamental cubic.

The converse is true.

THEOREM. *Every rational covariant curve of the third order is a member of the pencil or sheaf which contains the fundamental cubic and its Hessian.*

PROOF. The equation of the cubic can be put into Hesse's normal form, by a proper choice of reference system,

$$x_1^3 + x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0.$$

This equation, and therefore also every rational covariant,* is symmetrical in x_1, x_2, x_3. That is, every covariant cubic K must be a linear combination of three definite sets of terms:

$$K = c_1(x_1^3 + x_2^3 + x_3^3) + c_2x_1x_2x_3 + c_3(x_1^2x_2 + x_2^2x_3 + x_3^2x_1 + x_1^2x_3 + x_2^2x_1 + x_3^2x_2).$$

But the equations of f and H contain sets of terms like the first and second of these, with coefficients in general not proportional. Therefore a linear combination, K' of K, f, and H, can be formed, a rational covariant, which shall contain only the third set of terms

$$K' = K + l_1 \cdot f + l_2 \cdot H = l_3(x_1^2x_2 + x_2^2x_3 + \text{etc.}).$$

It is now to be shown that $l_3 = 0$. If not, *we should have a covariant cubic curve*

$$K' = 0$$

which contains all the vertices of our triangle of reference. This triangle is any one of the four inflexional triangles of the fundamental cubic. Therefore the twelve vertices of those four triangles would lie on the same curve of the third order. But we have seen (IV, § 3) that those 12 vertices are so placed that a line joining any two not in the same triangle will contain also two other vertices, one from each of the remaining triangles. Consequently there are four ver-

* Because every permutation of the coordinates x_1, x_2, x_3 is a linear transformation which does not alter the fundamental cubic.

tices on some lines; but no three lines contain all twelve; and so *no cubic curve, not even a degenerate cubic, can contain them all.* That is, there is no such covariant cubic as our hypothesis would require, hence l_3 must be zero. Accordingly we have

$$K + l_1 f + l_2 H = 0,$$

so that K, any covariant cubic, is a linear combination of f and H. q. e. d.

§ 2. Covariant curves of the third class.

The Hessian normal form of the cubic is not altered by any of the six possible permutations of the vertices of the triangle of reference. Hence it follows, just as in the case (§ 1) of covariants of the third order, that *any covariant of the third* class, L, must be a linear combination of three definite symmetric sets of terms:

$$L = l_1(u_1^3 + u_2^3 + u_3^3) + l_2 \cdot u_1 u_2 u_3 + l_3 \cdot (u_1^2 u_2 + u_2^2 u_3 + u_3^2 u_1 + u_1^2 u_3 + u_2^2 u_1 + u_3^2 u_2).$$

We know already one such covariant, the Cayleyan (III, § 8), which contains only the first and second of these three sets. If we can find a second, the argument of § 1 can be repeated here to show that any third is a linear combination of the first two.

There are two convenient ways of deriving a new covariant of the third order, both of which we shall explain. The first consists in calculating the Cayleyan covariant, $\sum'(u)$ or $\sum'$, for the indeterminate curve: $f + \lambda H = 0$ of the syzygetic pencil. As $\sum$ is of the third degree in coefficients of its fundamental form, it would be of degree 3 in the parameter λ:

$$\sum'_{f+\lambda H} = \sum'_f + \lambda \cdot \sum' + \lambda^2 \sum'' + \lambda^3 \sum^{(3)}.$$

Since linear transformation of coordinates does not affect λ, the terms in λ, λ^2, λ^3 must themselves be covariant separately.

We shall calculate explicitly only Σ'; since that will prove to be independent of Σ, and will supply the needed evidence that all others of the third class, including Σ'' and $\Sigma^{(3)}$, are compounded linearly of the first two, Σ and Σ'.

It will make clearer the explanation of both this and the second method if we take the fundamental cubic in a modified form, with four constants instead of the single m that occurs in Hesse's normal forms. Let

$$f = a_1x_1^3 + a_2x_2^3 + a_3x_3^3 - 6bx_1x_2x_3,$$

and

$$H = \alpha_1x_1^3 + \alpha_2x_2^3 + \alpha_3x_3^3 - 6\beta x_1x_2x_3,$$

where

$$\alpha_1 = a_1b^2, \quad \alpha_2 = a_2b^2, \quad \alpha_3 = a_3b^2,$$

$$\beta = \frac{1}{6}(a_1a_2a_3 - 2b^3).$$

The Cayleyan is found for this set of coefficients, just as for Hesse's form in Chapter III, § 8, by expanding a six-rowed determinant. It is

$$\Sigma_f = \Sigma = -2b(a_2a_3u_1^3 + a_3a_1u_2^3 + a_1a_2u_3^3) + (2a_1a_2a_3 + 8b^3)u_1u_2u_3.$$

Apply Taylor's formula to develop into a series the expression $\Sigma_{f+\lambda H}$ above. This requires for the calculation of Σ' the operation

$$\left(\alpha_1\frac{\partial}{\partial a_1} + \alpha_2\frac{\partial}{\partial a_2} + \alpha_3\frac{\partial}{\partial a_3} + \beta\frac{\partial}{\partial b}\right)\Sigma = \Sigma',$$

and this is found to be

$$\Sigma' = -\frac{1}{3}(a_1a_2a_3 + 10b^3)(a_2a_3u_1^3 + a_3a_1u_2^3 + a_1a_2u_3^3) + 2b^2(5a_1a_2a_3 - 4b^3)u_1u_2u_3.$$

This is a covariant of the kind required for the foregoing argument. For Hesse's normal form it reduces to the following:

$$-\frac{1+10m^3}{3}(u_1^3+u_2^3+u_3^3)$$
$$+2m^2(5-4m^3)u_1u_2u_3,$$

and its coefficients are evidently not proportional to those of the Cayleyan except when $8m^3-1=0$, the case of degenerate curves.

The second method is the so called method of Evectants. The expression $(u_1x_1+u_2x_2+u_3x_3)$, or (ux), is an *identical covariant*; and any invariant may be constructed for the indeterminate cubic $f+\lambda(ux)^3$ instead of f alone, giving rise to various covariants. Thus both invariants of the cubic, S, and T, will give expansions in powers of λ:

$$S_{f+\lambda(ux)^3}=S+\lambda S'+\text{ etc.}$$
$$T_{f+\lambda(ux)^3}=T+\lambda T'+\text{ etc.}$$

These expressions S' and T' will be covariants of the third class, calculated by Taylor's formula,

$$S'=\left(u_1^3\frac{\partial}{\partial a_1}+u_2^3\frac{\partial}{\partial a_2}+u_3^3\frac{\partial}{\partial a_3}-u_1u_2u_3\frac{\partial}{\partial b}\right)S.$$
$$T'=\left(u_1^3\frac{\partial}{\partial a_1}+\cdots\right)T.$$

This process of forming covariants from invariants is known as *evection*, and the resulting forms are evectants. In strictness, the evection process should include partial differentiation with respect to all ten constants of a general fundamental cubic, yielding terms in $u_1^2u_2$, etc., if such terms would not vanish identically. But as we have seen already that no such terms can occur in covariants of this normal cubic, this omission leads to no error, and our four-term operator gives the complete form.

Since for this particular cubic the invariants are:

$$S = b(a_1a_2a_3 + b^3)$$
$$T = a_1^2a_2^2a_3^2 + 20b^3a_1a_2a_3 - 8b^6,$$

the corresponding evectants are:

$$S' = b(a_2a_3u_1^3 + a_3a_1u_2^3 + a_1a_2u_3^3) - (a_1a_2a_3 + 4b^3)u_1u_2u_3;$$

$$T' = 2(a_1a_2a_3 + 10b^3)(a_2a_3u_1^3 + a_3a_1u_2^3 + a_1a_2u_3^3) - 12b^2(5a_1a_2a_3 - 4b^3)u_1u_2u_3.$$

These two are not new, since comparison with $\sum$ (the Cayleyan) and its derivative by the first method, $\sum'$, show that

$$S' = -\tfrac{1}{2}\sum,$$
$$T' = -6\sum'.$$

Invariants can be derived from these cubic covariants by the reverse process. We see at once that the substitution of terms in u_1, u_2, u_3, for coefficients a_1, a_2, a_3 and b effected by evection can be restored by the operation

$$\left(a_1\frac{\partial^3}{\partial u_1^3} + a_2\frac{\partial^3}{\partial u_2^3} + a_3\frac{\partial^3}{\partial u_3^3} - b\frac{\partial^3}{\partial u_1\partial u_2\partial u_3}\right), \text{ or } \Omega'$$

Also that

$$\Omega' S' = 4S, \quad \Omega' T' = 6T.$$

Notice also, what will soon be required for use:

$$\Omega'(6TS' - 4ST') \equiv 0.$$

The curve of third class, $6TS' - 4ST' = 0$, is the unique curve in the pencil

$$S' + kT' = 0$$

which has its left hand member reduced to zero by the Ω' process. We now know also that this pencil comprises all rational covariant curves of the third class, accordingly we have the equation of the only rational covariant curve of the third class which is reduced to an identity by the Ω' process. Such curves have received a special class-name: they are said to be *apolar* to the fundamental cubic.

By the Ω' process we can derive rational invariants from covariants of the third class, and conversely by the inverse process. Since therefore all rational covariants of the third class are rationally expressed as linear combinations of S' and T', *all rational invariants are rational functions of* S *and* T.

§ 3. The doubly apolar curve of the third class.

Two curves, one of the third order: $f(x) = 0$, the other of the third class: $\varphi(u) = 0$, are called mutually *apolar* when their equations satisfy a certain condition, bilinear in the coefficients. Write the equations symbolically

$$f(x) = (ax)^3 = 0, \quad \varphi(u) = (\alpha u)^3 = 0$$

then the condition of apolarity is

$$(a\alpha)^3 = 0.$$

A more involved relation is *double apolarity*, when every conic polar of the one is apolar to every conic polar of the other:

$$(ax)(\alpha u)(a\alpha)^2 = 0,$$

an identity in the variables (x) and (u). Double apolarity implies simple apolarity, as is evident from the conditions, but the converse is not true. The former is nine conditions, the latter only one. We shall prove:

THEOREM. *To a given non-singular plane curve of the third order there is one, and only one, doubly apolar curve of the third class*; *and its equation is the rational covariant of the given cubic,*

$$\Pi \quad 6TS' - 4ST' = 0.$$

PROOF. Take the Hessian normal equation modified as in § 2

$$f(x) = a_1x_1^3 + a_2x_2^3 + a_3x_3^3 - 6bx_1x_2x_3 = 0,$$

and write the other equation with its ten terms

$$\varphi(u) = \sum \pi_1 u_1^3 + 3\sum (p_1 u_1^2 u_2 + q_1 u_1 u_2^2) + 6\pi_{123} u_1 u_2 u_3 = 0.$$

For double apolarity nine terms are required to vanish. Three which contain x_1 are these:

$$\begin{aligned}(a_1\pi_1 - 2b\pi_{123})u_1 \\ + (a_1p_1 - 2bp_2)u_2 \\ + (a_1q_3 - 2bq_2)u_3.\end{aligned}$$

Permutation of indices 1, 2, 3, give 6 more terms, and all 9 must vanish. The set in p_1, p_2, p_3 determine $p_1 = p_2 = p_3 = 0$, except for $8b^3 = a_1a_2a_3$; similarly we find $q_1 = q_2 = q_3 = 0$ except for cases where $8b^3 = a_1a_2a_3$. Finally the other 3 equations give the ratios:

$$\pi_{123} : \pi_1 : \pi_2 : \pi_3$$
$$\frac{1}{2b} : \frac{1}{a_1} : \frac{1}{a_2} : \frac{1}{a_3}.$$

We shall avoid fractions by adopting, as normal form for the equation of this curve,

$$\prod(u) = 2b(a_2a_3u_1^3 + a_3a_1u_2^3 + a_1a_2u_3^3) + 6a_1a_2a_3u_1u_2u_3 = 0.$$

On comparison with the expression

$$6TS' - 4ST'$$

this is found to lack only the factor $-(a_1a_2a_3 - 8b^3)^2$. *The curve is completely defined by the requirement of double apolarity and is also completely defined by the requirement that*

its equation shall be a rational covariant together with that of simple apolarity.

Among other features, this curve of the third class is of value geometrically, giving a complete duality to the theory of the cubic.

THEOREM: *The Hessian of the curve* $\prod(u) = 0$ *is the Cayleyan of the fundamental cubic.* That is, the covariant whose equation is formed from the line-equation $\prod(u) = 0$ in the same way as the Hessian is formed from the point-equation $f(x) = 0$ is found to be the Cayleyan. To prove this, it is necessary only to use the coefficients of the $\prod(u)$ equation in place of those in the Hessian equation, and this substitution, after cancellation of a factor $2a_1^2a_2^2a_3^2$, yields the Cayleyan.

Remember now that the Cayleyan was first defined as the envelope of the lines that join corresponding or conjugate points on the Hessian curve; and that these lines, in pairs, intersect on the Hessian, a pair forming a degenerate polar of the fundamental cubic. Duality, the mutual relation of apolarity between conic polars of $\prod(u)$ and conic polars of $f(x)$, gives this theorem.

THEOREM. *Quadric polars with respect to* $\prod(u)$ *of lines which touch the Cayleyan are degenerate, consisting of two points on the Hessian. The line joining two such points meets the line, whose polar they are, in another point of the Hessian, the two lines constituting the degenerate conic polar, with respect to* $f(x)$, *of some fourth point on the Hessian.*

THEOREM. *If a line* l *touches the Cayleyan, its polar with respect to the line-cubic,* $\prod(u) = 0$, *is a degenerate line-quadric consisting of two points on the Hessian. Those points are a conjugate pair, and the line* l' *which joins them is a second tangent to the Cayleyan, conjugate to* l *with respect to the curve* $\prod(u) = 0$. *In short, as conjugate points of the Hessian lie on a tangent of the Cayleyan, so conjugate lines of the Cayleyan (with respect to* $\prod(u) = 0$) *meet on the Hessian.*

As the set of four points where tangents from a point of the Hessian touch that curve, so the dual figure for the Cayleyan is noteworthy.

Finally, as all covariant curves of the third order have the

same set of 9 inflexional points, so all covariant curves of the third class have the same 9 inflexional tangents, — the harmonic polars of those points, each containing 4 vertices of inflexional triangles.

§ 4. The cubic curve whose first polars are a given net of conics.

In Chapter VI, § 8 we examined the question whether, a net of conics being given, we could determine completely a cubic of which they should be the conic polars. When such a cubic exists, its algebraic determination is made more symmetrical by the notion of apolarity and the use of the line-cubic $\prod(u) = 0$ as an auxiliary.

The procedure will be this. Given three conics whose equations are, written symbolically, $(ax)^2 = 0$, $(bx)^2 = 0$, and $(cx)^2 = 0$. First determine the cubic line-equation $\prod(u) = (\pi u)^3 = 0$ by the 3 identical equations of apolarity

$$(a\pi)^2(\pi u) = 0,\ (b\pi)^2(\pi u) = 0,\ (c\pi)^2(\pi u) = 0.$$

These 9 equations are linear in terms of the π_{ijk}, and give a unique determination, save in the exceptional case noted in Chapter VI.

Now determine in the same way the curve $f(x) = (fx)^3 = 0$ by the requirement of *double apolarity* to $\prod(u)$:

$$(\pi f)^2(\pi u)(fx) = 0.$$

These 9 equations give uniquely, since they are linear, the coefficients f_{ijk}, which were to be found. This rule is most conveniently and briefly expressed as a theorem.

Theorem. *When two cubics, the one a point-locus, the other a line-locus, are doubly apolar to to each other, then all the conics apolar to either cubic are conic polars of points or lines with respect to the other cubic. From any three independent conic polars of either cubic, both cubics, if determinate, can be found by solving systems of linear equations.*

Chapter XI.

Simple apolarity between a curve of the third order and a curve of the third class.

§ 1. Definitions and a lemma on linear systems.

The conic polars of points of a plane with respect to a cubic in that plane we have called for brevity its conic polars. In this chapter it will be convenient to use the term apolar conics; and usually the cubic implied will be a line-locus, so that the apolar conics will be point-loci.

Apolarity is a bilinear relation, linear in the coefficients of each of two related curves. Hence if two point-cubics, $f_1 = 0$ and $f_2 = 0$, are apolar to a line cubic $\varphi = 0$, or K, any linear compound $f_1 + \lambda f_2 = 0$ is also apolar to K. The equation of a cubic, unrestricted, contains linearly ten homogeneous parameters. If it is apolar to a given line-cubic K, it can be fully represented by nine such parameters; if it is apolar to r line-cubics, by $10 - r$, provided that those r line-cubics have linearly independent equations. An easy extension of this argument will give the theorem following.

Theorem 1. *If r point-cubics are apolar to s line-cubics, and $r + s = 11$, then either the first set or the second set have equations connected by a linear identity.*

Closely connected with the theory of apolarity is that of polar quadrangles, polar pentagons, etc. A line-cubic K, whose equation is $\varphi(u) = 0$, has a polar quadrangle if its equation is expressible by linear combination of the cubes of four equations of single points. Thus $u_1\alpha_1 + u_2\alpha_2 + u_3\alpha_3 = 0$, or $(u\alpha) = 0$, is the equation of a point. Let $\alpha, \beta, \gamma, \delta$ be four points, then the equation

$$k_1(u\alpha)^3 + k_2(u\beta)^3 + k_3(u\gamma)^3 + k_4(u\delta)^3 = 0$$

represents a cubic having those four points for a polar quadrangle.

If a cubic $f(x) = 0$ passes through the point α, it is apolar to that point triply-counted. For the condition that must be satisfied by the two equations

$$f(x) = 0 \quad \text{and} \quad (u\alpha)^3 = 0$$

is precisely $f(\alpha) = 0$. Hence further

THEOREM 2. *If a line-cubic K has four points $\alpha, \beta, \gamma, \delta$, for a polar quadrangle, it is apolar to all point-cubics of the six-parameter family that contain those four points. Conversely, if K is apolar to six linearly independent point-cubics through four distinct points $\alpha, \beta, \gamma, \delta$, it has those four points for a polar quadrangle.* In proof of the latter, the converse part of this statement, the hypothesis gives us, in the notation of Theorem 1, $r = 6$; and there are beside K itself four other line-cubics apolar to those six, namely the triply-counted points $(u\alpha)^3 = 0$, $(u\beta)^3 = 0$, $(u\gamma)^3 = 0$, and $(u\delta)^3 = 0$, so that $s = 5$, and $r + s = 11$. But the point-cubics are linearly independent by hypothesis, therefore there is at least one identical linear relation

$$\lambda_1 K + \lambda_2 (u\alpha)^3 + \lambda_3 (u\beta)^3 + \lambda_4 (u\gamma)^3 + \lambda_5 (u\delta)^3 \equiv 0,$$

and in this, λ_1, cannot be zero, since there is no linear identity among the cubes of fewer than 5 linear forms. Hence the equation of K is reducible to a sum of cubes of the four point-equations, as was affirmed by the theorem.

THEOREM 3. *If a line-cubic K has four points $\alpha, \beta, \gamma, \delta$ for a polar quadrangle, it is apolar to every conic through those four points.*

The proof is the same as for the direct part of Theorem 2. Also the converse is true.

THEOREM 4. *Conversely, the four intersections of two apolar conics constitute a polar quadrangle of the line-cubic K.*

PROOF. This follows from Theorem 1 if we can discover 6 linearly independent point-cubics through the 4 points. Those may be degenerate. Let the equations of the conics be $S_1 = 0$ and $S_2 = 0$; and $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, be any three lines not meeting on either conic, nor copunctual.

The cubics:

$$S_1x_1 = 0, \quad S_1x_2 = 0, \quad S_1x_3 = 0,$$
$$S_2x_1 = 0, \quad S_2x_2 = 0, \quad S_2x_3 = 0,$$

are not linearly dependent. If they were, we should have an identity of the form

$$S_1 \cdot (l_1x_1 + l_2x_2 + l_3x_3) \equiv S_2 \cdot (m_1x_1 + m_2x_2 + m_3x_3),$$

Then if the lines (l) and (m) are different, (l) must constitute a part of the conic S_2, and (m) a part of S_1, the residual parts also must be identical. This would make the Cayleyan of K degenerate: we may exclude that case by appending to the theorem the condition: *if no three of those four intersections are collinear.* With this amended hypothesis, the theorem is proven.

We have seen that the polar quadrangles of a cubic K are the intersections of pairs in its net of apolar conics. A point-cubic may contain one such set of 4 points, and is then apolar to K. But it cannot contain two polar quadrangles unless it is degenerate.

THEOREM 5. *A proper point-cubic may contain one set of* 4 *points that constitute a polar quadrangle of a line-cubic* K; *or it may contain no such quadrangle*; *but it cannot contain two polar quadrangles of* K.

PROOF. The first part is obvious from the preceding. The last part follows from Theorem 3 and the fact that apolar conics of K constitute a linear net, so that the 8 points where two pairs intersect must lie on another conic of the net, too many for the common points of a conic and a proper cubic. As to the remaining part of the theorem, the second, it is evident from the existence of point cubics not apolar to K.

THEOREM 6. *There are apolar cubics which do not contain any polar quadrangle.*

PROOF. The counting of constants in a cubic containing a polar quadrangle shows only 7, instead of the 8 required for an apolar cubic. If $\varphi_1 = 0$ and $\varphi_2 = 0$ are two apolar

conics, each involving 2 parameters, and $l_1 = 0$ and $l_2 = 0$ any two lines, the equation $l_1\varphi_1 - l_2\varphi_2 = 0$ has apparently 9 constants. But only 7 are available, since 2 must remain indeterminate; for any other two conics of the pencil containing φ_1 and φ_2 may equally well be selected, and lines l_1', l_2' to correspond.

§ 2. Degenerate apolar curves. Polar Pentagons. Polar projectivity between pencils of lines and of conics.

A conic C and a line l together form a degenerate cubic point-locus, and this may be apolar to a line-cubic K. If we think of the conic as given, and also the cubic K, then the requirement of apolarity imposes a single condition upon the line l, a linear condition upon its coordinates; consequently all lines satisfying that condition pass through a common point. That point is called the polar point of the conic C with respect to the cubic K.

If five points $\alpha, \beta, \gamma, \delta, \varepsilon$, constitute a polar pentagon of the cubic K, take for conic C the two lines $\alpha\beta$ and $\gamma\delta$. The coordinates of those lines, v and w, are minors from the arrays

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} \text{ and } \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{vmatrix}.$$

As the cubic K has an equation of the form

$$k_1(u\alpha)^3 + k_2(u\beta)^3 + k_3(u\gamma)^3 + k_4(u\delta)^3 + k_5(u\varepsilon)^3 = 0,$$

the polar point of the degenerate conic vw will have an equation of 5 terms, the first of which is

$$k_1(v\alpha)(w\alpha)(u\alpha),$$

or, after substituting the values of coordinates v and w from the above arrays,

$$k_1(\alpha\beta\alpha)(\gamma\delta\alpha)(u\alpha).$$

Since one factor of this term, $(\alpha\beta\alpha)$, is identically zero, the

term vanishes. So with three other terms; the last term then gives the complete equation of the polar point:

$$k_5(\alpha\beta\varepsilon)(\gamma\delta\varepsilon)(u\varepsilon) = 0.$$

But this is the equation of the fifth point, e, of the pentagon, unless that point is on either the line $\alpha\beta$ or the line $\gamma\delta$. Exclude this case, which is really not exceptional. A second pair of lines, $\alpha\gamma$ and $\beta\delta$, would constitute a conic having the same polar point ε; so also $\alpha\delta$ and $\beta\gamma$. Now any conic through the four points $\alpha, \beta, \gamma, \delta$ has its equation linearly compounded of two of these degenerate equations. Hence the following theorem.

THEOREM 1. *If 5 points constitute a polar pentagon of a line-cubic K, then all conics passing through four of those points will have as their common polar point with respect to K the fifth point of the pentagon. Any conic through four of those points, with any line through the fifth point, will constitute a point-cubic apolar to K.*

Notice that in this theorem we have two pencils, one of conics through α, β, γ, and δ, the other of lines through ε. Relate these pencils projectively, and the locus of intersections of corresponding elements will be a point-cubic, necessarily containing all five base-points, and therefore apolar to the cubic K.

Consider next any pencil of conics whose base-points are not four points of a polar pentagon. Each conic of the pencil has a determinate polar point, and those points form a line projectively related, point to conic, with the pencil of conics. Call that projective relation of the pencils, the *polar projectivity* with respect to the line-cubic K.

In the general case, the polar projectivity of the two pencils will be one-to-one. There are however certain special cases.

a) The pencil of conics lies wholly in the net of apolar conics. The pole of every conic in the pencil is indeterminate. All projective relations of the two pencils are polar.

b) There is in the pencil a single apolar conic. Then the others have all the same polar point. Unless that point is

the center of the pencil of lines, the one line containing that point corresponds in the polar projectivity to all conics of the pencil.

If namely a conic $(k_1x)^2 = 0$ is apolar, and $(k_2x)^2 = 0$ any other in the pencil, then the polar point of the general conic

$$(k_1x)^2 + \lambda(k_2x)^2 = 0$$

is given by

$$(k_1\alpha)^2(\alpha u) + \lambda(k_2\alpha)^2(\alpha u).$$

But the apolarity hypothesis is

$$(k_1\alpha)^2(\alpha u) = 0.$$

Hence the pole is given by

$$\lambda(k_2\alpha)^2(\alpha u) = 0$$

the same for all values of λ.

c) Four points on one conic, with the center of the pencil of lines, constitute a polar pentagon. To that one conic the corresponding line is any line of the pencil.

It is necessary next to find what projectivities between a pencil of lines and a pencil of conics will generate an apolar cubic, in the general case and also in each of these special cases.

§ 3. Condition that projective pencils of lines and of conics may generate an apolar cubic.

If a non-singular cubic is generated by the method of Chasles (VII, § 1) by a pencil of conics and a projective pencil of lines, the projectivity is one-to-one with no exceptional lines or conics. Whether the resulting cubic shall be apolar, or not, depends on the relation of the generating projectivity to the polar projectivity discussed above. If the latter also is entirely regular, we have this theorem:

THEOREM 1. *The cubic generated by the two pencils is apolar to the line-cubic K if the generating projectivity of the pencils is in involution with their polar projectivity.*

PROOF. Denote the pencils by the symbolic equations:

$$\lambda_1(px) + \lambda_2(qx) = 0, \tag{1}$$

$$\mu_1(lx)^2 + \mu_2(mx)^2 = 0, \tag{2}$$

and the generating projectivity by the equation:

$$A\lambda_1\mu_1 + B\lambda_2\mu_1 + C\lambda_1\mu_2 + D\lambda_2\mu_2 = 0; \tag{3}$$

also the polar projectivity by the statement that line (1) and conic (2) form a degenerate cubic apolar to K. the conic whose line-equation is $(u\alpha)^3 = 0$. This statement is expressed by the equation:

$$[\lambda_1(p\alpha) + \lambda_2(q\alpha)][\mu_1'(l\alpha)^2 + \mu_2'(m\alpha)^2] = 0. \tag{4}$$

The involution condition is the requirement that the relation of two variable points (μ_1, μ_2) and $(\mu_1' : \mu_2')$ having the same (λ) in (3) and (4) shall be symmetrical; i. e. that

$$A(q\alpha)(m\alpha)^2 - B(p\alpha)(m\alpha)^2 - C(q\alpha)(l\alpha)^2 + D(p\alpha)(l\alpha)^2 = 0. \tag{5}$$

It is to be shown that this same relation is the condition that the cubic generated by pencils (1) and (2) in relation (3) shall be apolar to the cubic K, or $(u\alpha)^3 = 0$. That cubic has the equation:

$$A(qx)(mx)^2 - B(px)(mx)^2 - C(qx)(lx)^2 + D(px)(lx)^2 = 0.$$

It is apolar to K if the relation is satisfied:

$$A(q\alpha)(m\alpha)^2 - B(p\alpha)(m\alpha)^2 - C(q\alpha)(l\alpha)^2 + D(p\alpha)(l\alpha)^2 = 0. \tag{6}$$

But relations (5) and (6) are identical. q. e. d.

Next we consider in their order the special kinds of polar projectivity, a), b), and c).

Case a. Both conics, $(mx)^2 = 0$ and $(lx)^2 = 0$ are apolar to K; that is, symbolically, every expression containing a factor $(m\alpha)^2$ or $(l\alpha)^2$, is zero. Hence *equation* (6) *is satisfied*

for all values of A, B, C, and D; and every projective relation of the two pencils generates an apolar cubic.

Case b. When there is one apolar conic in the pencil, it corresponds in the polar projectivity to all lines of the plane and therefore to all those in the pencil of lines. If the common pole of all the conics is the center of the pencil of lines, the case is the same as the foregoing case (*a*); but if not, then only that line of the pencil which contains the common pole will correspond to all conics of the pencil.

For the generating projectivity, relation (5) or (6) gives the necessary and sufficient condition. If $(px) = 0$ is the exceptional line, and $(lx)^2 = 0$ the apolar conic, terms of (5) involving either $(p\alpha)$ or $(l\alpha)^2$ will vanish, leaving $A = 0$. Accordingly the generating projectivity (3) gives $\lambda_2 = 0$ and $\mu_2 = 0$ as corresponding elements in the two pencils, that is, the exceptional elements. Conversely, if these two correspond in (3), A must be zero, and condition (5) or (6) is satisfied, and the resulting cubic is apolar.

Case c. Since one exceptional conic corresponds, in the polar projectivity, to all lines of the pencil, there must be also an exceptional line, to which correspond all conics of the pencil. From these, the argument proceeds exactly as in the preceding case (*b*).

THEOREM 2. *The cubic generated by a pencil of lines and a pencil of conics in projective relation is apolar in all cases where the polar projectivity of the two pencils is degenerate, provided the singular line and the singular conic in the latter form a pair in the generating projectivity. The converse also is true.*

From this results an important corollary:

THEOREM 3. *If an apolar cubic is cut by an apolar conic in six points, all conics that contain four of those points have their common polar point on the line which joins the remaining two points of intersection.* For the cubic can be generated by the conics through the four points and the lines through their corresidual single point, provided the apolar conic of our hypothesis is related, in the generating projectivity, to

the singular line of the pencil. But in that generation the line corresponding to the apolar conic is the line which joins the remaining two points; and that line, as being singular, must contain the common polar point of all conics in the pencil. q. e. d.

§ 4. O. Schlesinger's construction of an inscribed polar pentagon of a line-cubic K.*

THEOREM 1. *If a non-singular point-cubic C is apolar to a non-singular line-cubic K, through every point P of the former can be passed a conic whose five other intersections with the cubic C shall be a polar pentagon of the line-cubic K.*

PROOF. The Hessian of the line-cubic K is of the third class, hence only three of the lines through P will be constituent parts of degenerate apolar conics of K. Let a line which is not one of those three meet the cubic C in points P, P_1, P_2. Construct the one apolar conic which contains P_1 and P_2; let it intersect the apolar cubic C in four other points, A_1, A_2, A_3, A_4. Then by theorem 3, § 3, the polar point of all conics in the pencil through A_1, A_2, A_3, A_4 lies on the line PP_1P_2; call that polar point Q. It does not coincide with P unless the four A-points form with P a polar pentagon, in which case the entire pencil of conics would be apolar conics, and the A-points therefore a polar quadrangle. Reject this as an exceptional case (Theorem 6, § 1).

Through P and Q construct the apolar conic, and let it meet the cubic C further in the points B_1, B_2, B_3, B_4, B_5. *These five B-points are a polar pentagon of K.*

In the first place, the B-points and the A-points form a *complete set* of 9 points on the cubic C. For while the points P_1, P_2 are residual to P and to the 4 A-points, making the latter corresidual to P, the B-points are residual to P. Therefore by a theorem of Chapter VIII, § 2, the B-points are residual to the A-points, making a complete set of nine. q. e. d.

* This, and nearly all of this Chapter XI, are adapted from the article by O. Schlesinger in vol. 30 of the Mathematische Annalen, *Ueber conjugierte Curven etc.*, p. 453.

Secondly, *all the pencil of cubics through these* 9 *points are apolar to the line-cubic* K. For example, that cubic which contains the point Q, let it be called C_1, will have the point Q corresidual to the four A-points because it is residual to the five B-points. Hence C_1 is generated by the pencil of conics through the A-points and the pencil of lines through Q. But Q is the polar point of all conics in the pencil, and therefore (§ 3, Case *a*) the resulting cubic C_1 must be apolar, and since both C and C_1 are apolar, the same is true of all cubics in the pencil through the 9 points of their intersection.

Lastly we have through the five B-points five linearly independent point-cubics apolar to K. They are C, C_1, and three degenerate curves consisting of the *apolar* conic, which contains the B-points and P, and an arbitrary line. Denote the apolar conic by S_P, and any 3 independent lines by u, v, and w; then C, C_1, $S_P u$, $S_P v$, $S_P w$, are linearly independent cubics. Were they not, then must some cubic through the 9 A-points and B-points be degenerate, consisting of the conic S_P and some line r. The conic would contain 6 points of the set; the line r, 3. On the conic must lie therefore, beside the B-points, one of the A-points, say A_1, and this must coincide with P which is on both C and the conic S_P. Then will the apolar conic through the A-points contain not only points P_1 and P_2 but also P,—3 points of a line, and hence the whole line; whereas we stipulated in the construction that that line should not form part of any apolar conic.

Since then those five B-points, each triply counted, and the curve K are apolar to five linearly independent point-cubics, there is necessarily a linear identical relation between K and the cubes of those 5 point-equations; whence the points are a polar pentagon of K. q. e. d.

The conic S_P is uniquely determined when the point P is given, for it is the locus of the point Q on all secants of the cubic C through P. In proof of this, draw PQ arbitrarily, avoiding the 3 tangents from P to the Hessian of K. Let it meet the curve C again in P_1 and P_2. Then from these two points, determine the 4 A-points on C by the aid of the apolar conic through P_1 and P_2. Let the polar point of the

conics through A_1, A_2, A_3, A_4, be (on the line PQ) different, if possible, from Q; call it Q_1.

Upon this assumption, the line QQ_1 is a determinate line in the pencil about Q. That pencil, with the pencil of conics through the A-points, generates a cubic which contains Q and the 9 A-points and B-points, as before; and it will be apolar because it contains the 5 B-points. Therefore (§ 3, Case *b*) the line QQ_1 containing the common polar point Q_1 must be the line projectively corresponding to the apolar conic of the pencil, and the intersections of these two elements, P_1 and P_2, must lie on the cubic. That is to say, there is a cubic through Q containing the 9 points A and B of the cubic C, and also the points P_1 and P_2. Having thus 11 common points with C, it is identical with C. But it contains Q which does not lie upon C, an absurdity. Therefore Q_1 is not, as assumed, different from Q. All poles Q_1 lie therefore on the conic S_P, as was to be proven.

Notice further that any point B_1 of the polar pentagon is the polar point of all conics through the remaining four B-points (Theorem 1, § 2). Join B_1 and P, and produce the line to meet the cubic C again in a point P'. If now we repeat from P' the constructions made above from P, we shall find an apolar conic $S_{P'}$, through P' and B_1, whose five intersections with C aside from P' will constitute another polar pentagon. *Hence B_1 is a constituent point of at least two polar pentagons of K.* It is proven by O. Schlesinger, *loco citato*, that it belongs to no more than two.

Quite analogous to the theorem relating to mutually apolar conics is therefore the following beautiful theorem of Schlesinger: *If a point-cubic C is apolar to a line-cubic K, every point of the cubic C is residual to one set of five points on C constituting a polar pentagon of K, and is itself a member of two such inscribed polar pentagons of K.*

Appendix.

NOTE 1. (See Fig. 7, I, § 5.)

This is possible if there is only one real point at infinity on the odd branch. Any line can be projected to infinity, hence it will suffice if any one line can be proved to intersect the odd branch in only one real point. Suppose that an odd branch containing no double point is cut by every line through a fixed point P, not on the curve, in three distinct real points: we can show that this is absurd, i. e. that either the curve must have a double point or there must be at least one line meeting it in fewer than 3 points, hence in only one real point.

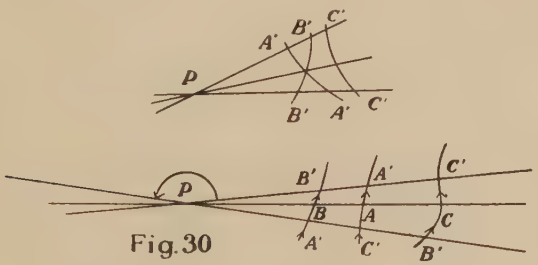

Fig. 30

PROOF. Let one fixed line through P meet the odd branch in $A\,B\,C$. Rotate a variable line from this position through 180°, i. e. until it first coincides with its former position. Call the varying points of intersection thereon A', B', C'. Now A', B', and C' describe during the rotation, by hypothesis, continuous real arcs, and must return to A, B, C taken in some order. If A', starting at A, returns to A, its path will be a closed branch not containing B or C, contrary to hypothesis. Let A' therefore return to B, B' to C, C' to A. On the varying line let P and A' at the initial position separate the pairs of points B, C. At the end of the rotation P and A' do not separate B', C'. During the rotation neither B' nor

C' can come to coincide with P, hence either B' or C' must have come to coincidence with A': if it were B', the real paths of A' and B' must cross at that point of coincidence, which is accordingly a real double point. As there is no real double point, by hypothesis, not every line through P can meet the odd branch in 3 real points. q. e. d.

Further inspection shows that a second double point would be required, whence the curve must be degenerate. Therefore even the cubic with double point or cusp is cut by some line in only one real point. The argument depends upon the fact that any possible cyclical permutation of $A'\ B'\ C'$ (here in the set of points P, A', B', C') requires at least two transpositions of one letter with another.—In the alternative, the actual case, that the transposition of points is effected at the passage from real to imaginary or conversely, this form of argument cannot be applied, for two imaginary points cannot separate two real points in a line.

NOTE 2. (See V, § 4, p. 75.)

From a point on an odd branch of a non-singular cubic only two real tangents can be drawn to that branch. Let the point be O. A line OP drawn to a second point P of the odd branch meets it in some third real point Q. As there are therefore lines through O cutting in three real points, by the preceding note there must be at least two tangents, and also a line OR cutting this branch in only one point, O. Let OP revolve about O first in one sense, then in the opposite; before reaching the position OR it must become at least once a tangent, say first in OA while revolving in the one sense, and in OB in the other sense. Then the odd branch must lie wholly in the

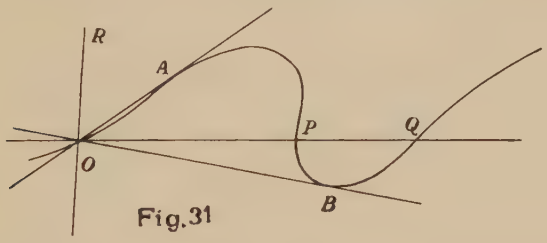

Fig.31

complete angle consisting of AOB and its vertical angle. For P and Q lie in that angle, and a point starting from Q to describe the branch does not cross the line OA at A, nor OB at B; but does cross both at O, since if it were tangent to either, as OA, at O, that line would meet the curve in two contacts or four points, and this is impossible. For the same reason the curve crosses the boundaries of this complete angle at no other point, and by construction there is no tangent to the curve in this angle. Hence there are only two tangents through O to the odd branch.

From a point on an even branch there are no tangents to the odd branch, since such a line would meet the even branch in at least two points, making, with the contact, four points of intersection.

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